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A Distributed Delay Model of One Ammensal on two Mutualistic Species

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Abstract: In this paper we study the dynamics of one ammensal and two species helping mutually one another. Here the first species(x) ammensal on the other two species (y, z), which are mutually helping one another. A distributed type delay is incorporated in the interaction of second and third species with the first species. Co-existing state of the system is identified and the stability is studied at this point. And numerical simulation is carried out to study the stability of the system using exponential type of kernel. We observed that the delay parameters improved the mutualistic species' densities.

Keywords: Ammensal, Mutualism , Co-existing state , stability, delay kernels**.**

1. Introduction:

Lokta [1] and Volterra [2] initiated research in theoretical ecology. Ecological interventions and stability analysis of species was addressed by May [3], Cushing [4] and Kapur [5, 6]. The interactions in ecology are mainly prey-predator, competitor, Ammensal and mutualistic relations.

Ammensalism is a relationship among the species, in which one species has adverse effect on another without any benefit. For example, Allelopathy emits or releases chemical substances which harms the neighbouring plants and herbs. Here Allelopathy harms the other species without any benefit or loss. And Mutualism is a relation, in which the interaction of two species gets mutual benefit from other. For example, cad fish and cleaner fish relation. Cleaner fish get food, by removing dead skin of cad fish.It is a fact that the time delay in ecological systems is a reality and it can have the complex effect on the dynamics of the system, i.e., loss of stability, induced oscillations and periodic solutions. Lakshmi Narayan et al.[8,9,10] investigated the prey – predator ecological models with a partical cover for the prey and alternative food for predator and time delay. Ravindra Reddy B et al.[11] studied "A Model of Two Mutually Interacting species with limited resources and a time delay. Paparao A V et al[13,15] discussed "A three species dynamical system of ammensal relationship of humans on plants and birds with time delay. Inspite of these model with one ammensal and two mutualistic species. The Distributed type of delay with exponential kernel was taken for investigation. The delay is incorporated in the interaction of the ammensal species with the two mutualistic species. The delay dynamics is studied with different kernel parameters a and λ. The delay kernel is significant in improving the mutual species population.

2. Mathematical Model:

The proposed mathematical model is a logistic growth model of three species with delay arguments, in which the first species(x) ammensal on the remaining two mutually helping species(y, z). Distributed type of delay is incorporated in the interaction of ammensal and the two mutual species. The model is considered by the system of integro-differential equations given by the following system of equations.

$$\frac{dx}{dt} = a_1 x\left[1 - \frac{x}{c_1}\right]$$

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$$\frac{dy}{dt} = a_2 y\left[1 - \frac{y}{c_2}\right] - \alpha_{21} y \int_{-\infty}^{t} w_1(t-u)\,x(u)du + \alpha_{23} yz \qquad (2.1)$$

$$\frac{dz}{dt} = a_3 z\left[1 - \frac{z}{c_3}\right] - \alpha_{31} z \int_{-\infty}^{t} w_2(t-u)\,x(u)du + \alpha_{32} zy$$

Where

x- is the Ammensal population,

y & z are the mutualistic species populations,

a_1, a_2 *and* a_3 are the growth rates of the ammensal and mutualistic species,

α_{21} & α_{31} are the decay rates of mutualistic species due to the attack of ammensal species. α_{23} & α_{32} are the growth rates of the mutualistic species due to the help from each other.

c_1, c_2 *and* c_3 are the carrying capacities of the ammensal and mutualistic species.

Here the variables *x*, *y*, *z* are non negative and the model parameters $a_1, a_2, a_3, \alpha_{21}, \alpha_{23}, \alpha_{31}$ and α_{32} are assumed such that they are non negative constants.

And taking $\frac{a_1}{c_1} = k_1, \frac{a_2}{c_2} = k_2, \frac{a_3}{c_3} = k_3$.

Putting t-u = s in (2.1), we get the following system of equations

$$\frac{dx}{dt} = a_1 x\left[1 - \frac{x}{c_1}\right]$$

$$\frac{dy}{dt} = a_2 y\left[1 - \frac{y}{c_2}\right] - \alpha_{21} y \int_0^{\infty} w_1(s)\,x(t-s)ds + \alpha_{23} yz \qquad (2.2)$$

$$\frac{dz}{dt} = a_3 z\left[1 - \frac{z}{c_3}\right] - \alpha_{31} z \int_0^{\infty} w_2(s)\,x(t-s)ds + \alpha_{32} zy$$

Choosing the kernels w_1 and w_2 such that

$$\int_0^{\infty} \boldsymbol{w_1(s)ds = 1}, \int_0^{\infty} \boldsymbol{w_2(s)ds = 1}, \int_0^{\infty} \boldsymbol{s\, w_1(s)ds} < \infty, \& \int_0^{\infty} \boldsymbol{s\, w_2(s)ds} < \infty \qquad (2.3)$$

3. Equilibrium Point:

The equilibrium points are obtained by solving $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0 \& \frac{dz}{dt} = 0$ (3.1)

For the system there are eight equilibrium points, out of those we considered only the co-existing state, which is given by

$$\bar{x} = c_1$$

$$\bar{y} = \frac{c_1(\alpha_{23}\alpha_{31} + k_3\alpha_{21}) - (a_3\alpha_{23} + a_2k_3)}{(\alpha_{23}\alpha_{32} - k_2k_3)}$$

$$\bar{z} = \frac{c_1(k_2\alpha_{31} + \alpha_{21}\alpha_{32}) - (a_3k_3 + a_2\alpha_{32})}{(\alpha_{23}\alpha_{32} - k_2k_3)}$$

This state would exist only when

$(c_1(\alpha_{23}\alpha_{31} + k_3\alpha_{21}) - (a_3\alpha_{23} + a_2k_3))(\alpha_{23}\alpha_{32} - k_2k_3) > 0$ and

$(c_1(k_2\alpha_{31} + \alpha_{21}\alpha_{32}) - (a_3k_3 + a_2\alpha_{32}))(\alpha_{23}\alpha_{32} - k_2k_3) > 0$ (3.2)

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4. Stability of the Co-existing State:

Theorem: The equilibrium point $(\bar{x}, \bar{y}, \bar{z})$ is locally asymptotically stable if $k_2\, k_3 > \alpha_{23}\, \alpha_{32}$

Proof: The variational matrix of the system is given by

$$J = \begin{bmatrix} -k_1\bar{x} & 0 & 0 \\ -\alpha_{21}\bar{y}w_1(s) & -k_2\bar{y} & \alpha_{23}\bar{y} \\ -\alpha_{31}\bar{z}w_2(s) & \alpha_{32}\bar{z} & -k_3\bar{z} \end{bmatrix} => A = \begin{bmatrix} k_1\bar{x} & 0 & 0 \\ \alpha_{21}\bar{y}w_1(s) & k_2\bar{y} & -\alpha_{23}\bar{y} \\ \alpha_{31}\bar{z}w_2(s) & -\alpha_{32}\bar{z} & k_3\bar{z} \end{bmatrix} \quad (4.1)$$

The characteristic equation of the $\boldsymbol{A}$ is $|A - \lambda I| = 0$

$$=> \lambda^3 + \lambda^2 b_1 + \lambda\, b_2 + b_3 = 0 \quad (4.2)$$

Where $b_1 = k_1\bar{x} + k_2\bar{y} + k_3\bar{z} > 0,$

$$b_2 = k_1\, k_2\bar{x}\bar{y} + k_1k_3\bar{x}\bar{z} + k_2\, k_3\bar{y}\bar{z} - \alpha_{23}\, \alpha_{32}\bar{y}\bar{z}$$

$$=> b_2 = k_1\, k_2\bar{x}\bar{y} + k_1k_3\bar{x}\bar{z} + (k_2\, k_3 - \alpha_{23}\, \alpha_{32})\bar{y}\bar{z} \quad \text{and}$$

$$b_3 = k_1\bar{x}(k_2\, k_3\bar{y}\bar{z} - \alpha_{23}\, \alpha_{32}\bar{y}\bar{z}) => b_3 = k_1\bar{x}\bar{y}\bar{z}(k_2\, k_3 - \alpha_{23}\, \alpha_{32})$$

$\Rightarrow b_1 b_2 - b_3 = (k_1\bar{x} + k_2\bar{y} + k_3\bar{z})(k_1\, k_2\bar{x}\bar{y} + k_1k_3\bar{x}\bar{z} + (k_2\, k_3 - \alpha_{23}\, \alpha_{32})\bar{y}\bar{z}) - k_1\bar{x}\bar{y}\bar{z}(k_2\, k_3 - \alpha_{23}\, \alpha_{32})$

$\Rightarrow \quad b_1 b_2 - b_3 = (k_1\bar{x} + k_2\bar{y} + k_3\bar{z})(k_1\, k_2\bar{x}\bar{y} + k_1k_3\bar{x}\bar{z}) + (k_2\bar{y} + k_3\bar{z})(k_2\, k_3 - \alpha_{23}\, \alpha_{32})\bar{y}\bar{z}$

$\Rightarrow \quad b_1 b_2 - b_3 > 0$ if $k_2\, k_3 > \alpha_{23}\alpha_{32}$ and

$\Rightarrow \quad b_3(b_1 b_2 - b_3) = k_1\bar{x}\bar{y}\bar{z}(k_2\, k_3 - \alpha_{23}\, \alpha_{32})((k_1\bar{x} + k_2\bar{y} + k_3\bar{z})(k_1\, k_2\bar{x}\bar{y} + k_1k_3\bar{x}\bar{z}) + (k_2\bar{y} + k_3\bar{z})(k_2\, k_3 - \alpha_{23}\, \alpha_{32})\bar{y}\bar{z})$

Which is positive if $k_2\, k_3 > \alpha_{23}\, \alpha_{32}$

We have $b_1 > 0$, $(b_1 b_2 - b_3) > 0$ and $b_3(b_1 b_2 - b_3) > 0$ if $k_2\, k_3 > \alpha_{23}\, \alpha_{32}$

Therefore, by Routh – Hurwitz criteria, the system is Asymptotically stable

if $k_2\, k_3 > \alpha_{23}\, \alpha_{32}$

Hence the interior equilibrium point $(\bar{x}, \bar{y}, \bar{z})$ is locally asymptotically stable

if $k_2\, k_3 > \alpha_{23}\, \alpha_{32}$

Let us define the kernels as follows $w_1(s) = w_2(s) = ae^{-as}\ for\ a > 0$, then the Laplace transform of $w_1(s) \& w_2(s)$ are defined as

$$\boldsymbol{w_1(\lambda) = w_2(\lambda) = \int_0^\infty e^{-\lambda t} a\, e^{-at} dt = \frac{a}{a + \lambda}}$$

Then the system is locally asymptotically stable if $k_2\, k_3 > \alpha_{23}\, \alpha_{32}$

5. Global Stability:

Statement: The co-existing state is globally asymptotically stable.

Proof: Let us consider the Lyapunov's function for the co-existing state is

$$V(\bar{x}, \bar{y}, \bar{z}) = \left\{x - \bar{x} - \bar{x}\ln\left(\frac{x}{\bar{x}}\right)\right\} + \left\{y - \bar{y} - \bar{y}\ln\left(\frac{y}{\bar{y}}\right)\right\} + \left\{z - \bar{z} - \bar{z}\ln\left(\frac{z}{\bar{z}}\right)\right\} \quad (5.1)$$

Here $\bar{x} \neq 0, \bar{y} \neq 0, \bar{z} \neq 0$

Differentiate (5.1) with respect to 't', we get

$$\frac{dV}{dt} = \left[\frac{x - \bar{x}}{x}\right]\frac{dx}{xt} + \left[\frac{y - \bar{y}}{y}\right]\frac{dy}{dt} + \left[\frac{z - \bar{z}}{z}\right]\frac{dz}{dt} \quad (5.2)$$

$$\Rightarrow \frac{dV}{dt} = \begin{Bmatrix} \left[\frac{x - \bar{x}}{x}\right](a_1 x - k_1 x^2) + \left[\frac{y - \bar{y}}{y}\right](a_2 y - k_2 y^2 - \alpha_{21} y x w_1(\lambda) + \alpha_{23} y z) \\ + \left[\frac{z - \bar{z}}{z}\right](a_3 z - k_3 z^2 - \alpha_{31} z x w_2(\lambda) + \alpha_{32} z y) \end{Bmatrix} \quad (5.3)$$

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$$\Rightarrow \frac{dV}{dt} = \left\{ \begin{array}{c} \left[\frac{x-\bar{x}}{x}\right](a_1 x - k_1 x^2) + \left[\frac{y-\bar{y}}{y}\right](a_2 y - k_2 y^2 - \alpha_{21} y \int_0^{\infty} w_1(s)\, x(t-s) ds + \alpha_{23} yz) \\ + \left[\frac{z-\bar{z}}{z}\right](a_3 z - k_3 z^2 - \alpha_{31} z \int_0^{\infty} w_2(s)\, x(t-s) ds + \alpha_{32} zy) \end{array} \right\}$$

$$\Rightarrow \frac{dV}{dt} = \left\{ \begin{array}{c} [x-\bar{x}](a_1 - k_1 x) + [y-\bar{y}](a_2 - k_2 y - \alpha_{21} \int_0^{\infty} w_1(s)\, x(t-s) ds + \alpha_{23} z) \\ +[z-\bar{z}](a_3 - k_3 z - \alpha_{31} \int_0^{\infty} w_2(s)\, x(t-s) ds + \alpha_{32} y) \end{array} \right\} \quad (5.4)$$

$$= -k_1(x-\bar{x})^2 + (y-\bar{y})[-\alpha_{21}(x-\bar{x}) - k_2(y-\bar{y})] + (z-\bar{z})[-\alpha_{31}(x-\bar{x}) - k_3(z-\bar{z})]$$

$$= -k_1(x-\bar{x})^2 - k_2(y-\bar{y})^2 - k_3(z-\bar{z})^2 - \alpha_{21}(x-\bar{x})(y-\bar{y}) - \alpha_{31}(x-\bar{x})(z-\bar{z}) \quad (5.5)$$

Using the basic inequality $ab \leq \frac{a^2+b^2}{2}$

$$= -k_1(x-\bar{x})^2 - k_2(y-\bar{y})^2 - k_3(z-\bar{z})^2 - \frac{\alpha_{21}}{2}[(x-\bar{x})^2 + (y-\bar{y})^2] - \frac{\alpha_{31}}{2}[(x-\bar{x})^2 + (z-\bar{z})^2] \quad (5.6)$$

$$= -(x-\bar{x})^2\left[k_1 + \frac{\alpha_{21}}{2} + \frac{\alpha_{31}}{2}\right] - (y-\bar{y})^2\left[k_2 + \frac{\alpha_{21}}{2}\right] - (z-\bar{z})^2\left[k_3 + \frac{\alpha_{31}}{2}\right] \quad (5.7)$$

$$\Rightarrow \frac{dV}{dt} < 0 \quad (5.8)$$

Hence the Normal steady state is globally asymptotically stable.

6. Numerical Examples:

S.No	Figures	Description
1	The figure (A)	Shows the change of *x*, *y* and *z* corresponding to time 't'.
2	The figure (B)	The Phase Portrait of *x*, *y* and *z*.

Example 1: a_1=5; a_2=5; a_3=5; α_{21}=0.05; α_{23}=0.05; α_{31}=0.05; α_{32}=0.05; c_1 =50; c_2 =150; c_3 =170;

The system of equations(2.2) is simulated using MATLAB (ode45). When the system without delay is solved, we get the following results. That is the system is neutrally stable to the fixed equilibrium point E(30.00, 91.55, 113.07) as illustrated in the graphs 6.1(A), 6.1(B) with the above parametric values.

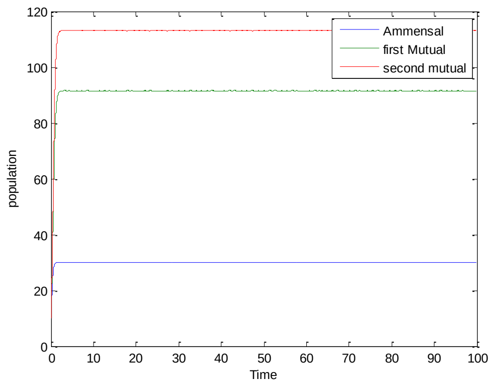

Fig:6. 1(A)

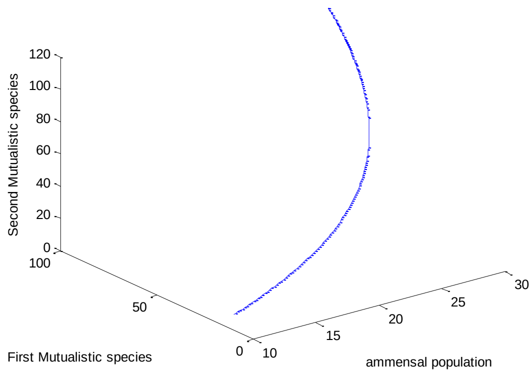

Fig: 6.1(B)

With the kernels as follows $w_1(s) = w_2(s) = ae^{-as}$ *for* $a > 0$, and the Laplace transform of $w_1(s) \& w_2(s)$ are defined as

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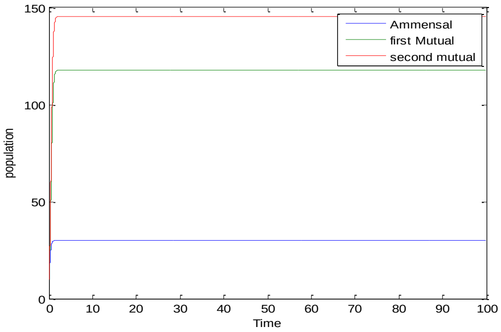
$$w_1(\lambda) = w_2(\lambda) = \int_0^\infty e^{-\lambda t} a\, e^{-at} dt = \frac{a}{a+\lambda}$$

The results, which are simulated with the parameters considered in example -1 and the specified delay parameters (a, λ) for the system of equations (2.2) shown as follows

1. a=0.1; λ=0.2; (30.00,117.70,145.38)



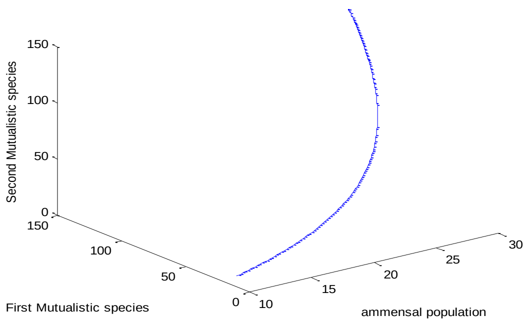
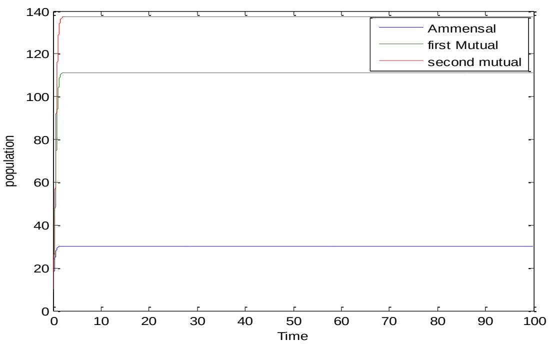


Fig:6. 1.1(A) Fig: 6.1.1(B)

The system is neutrally stable and due to the delay parameters, mutualistic species(second and third mutualistic species) population is increased from its initial growth when compared with no delay arguments are induced.

2. a=0.5; λ=0.5; (30.00,111.22,137.25)



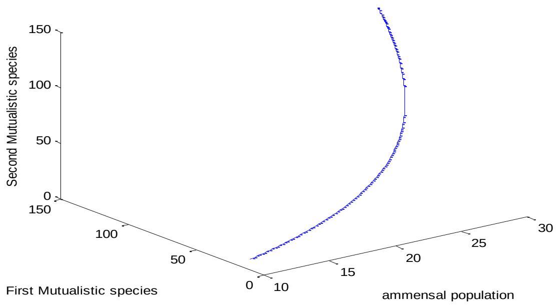


Fig:6. 1.2(A) Fig: 6.1.2(B)

The system is neutrally stable and due to the delay parameters, mutualistic species(second and third mutualistic species) population is increased from its initial growth when compared with no delay arguments are induced and are converges to the fixed equilibrium point E(30,111,137).

3. a=1; λ=5; (30.00,124.25,153.45)

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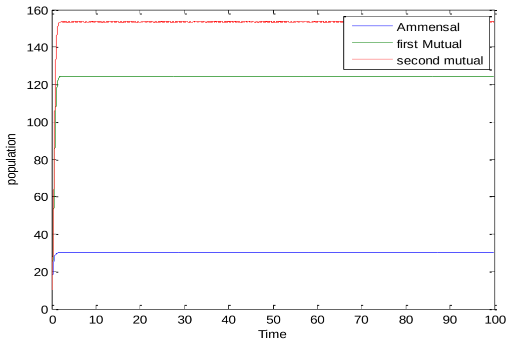


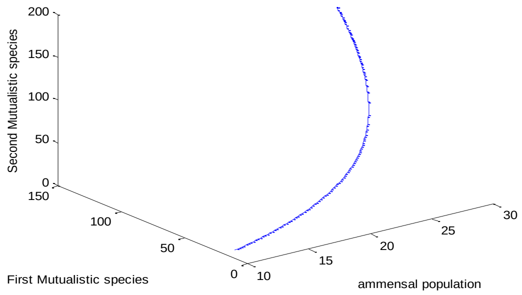
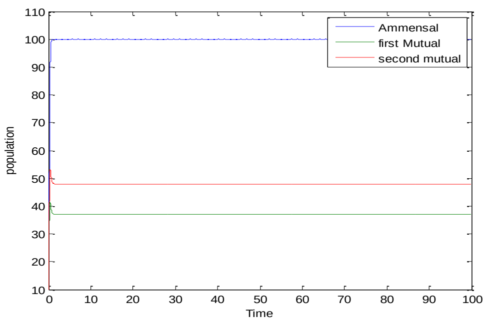


Fig:6. 1.3(A) Fig: 6.1.3(B)

The system is neutrally stable and due to the delay parameters, mutualistic species (second and third mutualistic species) population is increased from its initial growth when compared with no delay arguments are induced. And converges to the fixed equilibrium point E(30, 124, 153).

Example 2: a_1 =10; a_2 =10; a_3 =10; α_{21}=0.05; α_{23}=0.05;α_{31}=0.05;α_{32}=0.05;c_1 =100;c_2 =100; c_3 =100;(99.98, 36.98, 47.94)

Without delay parameters:



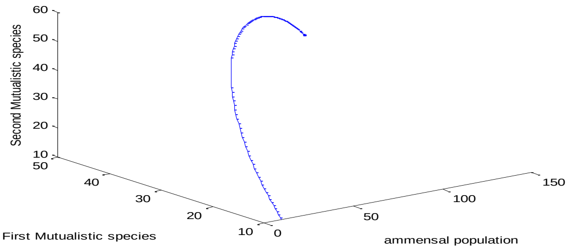
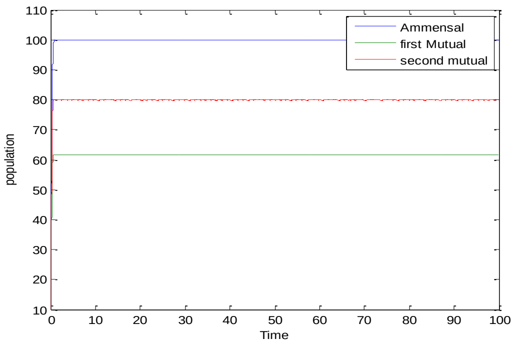


Fig:6.2(A) Fig: 6.2(B)

1. a=0.1; λ=0.2; (100.00,61.66,79.89)



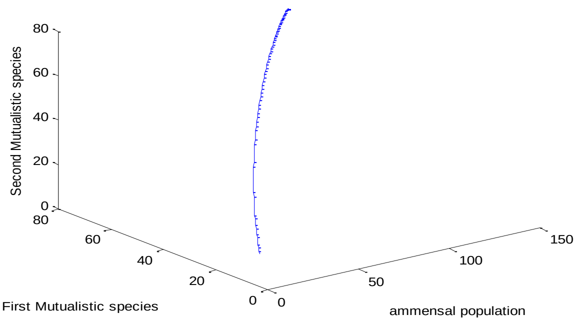


Fig:6.2.1(A) Fig: 6.2.1(B)

2. a=0.5; λ=0.5; (100.00,55.48,71.91)

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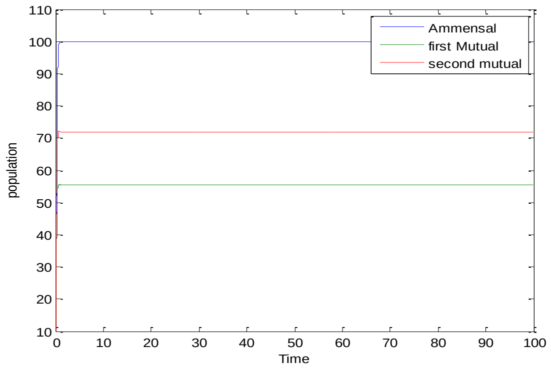

Fig:6.2.2(A)

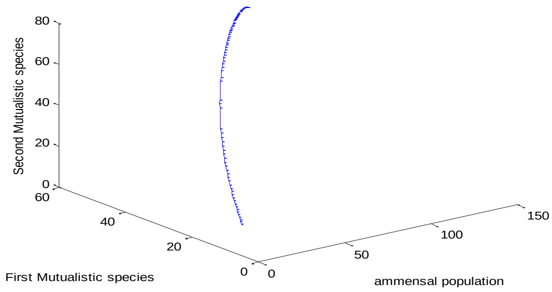

Fig: 6.2.2(B)

3. a=1; λ=5; (100.00, 67.81, 87.89)

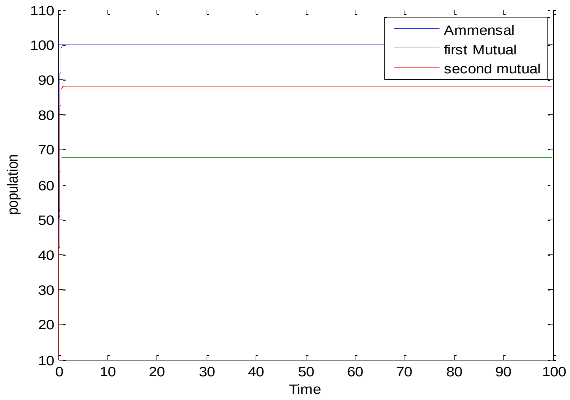

Fig:6.2.3(A)

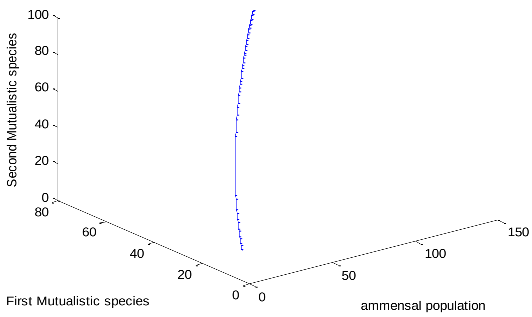

Fig: 6.2.3(B)

7. Conclusion:

We considered a three species syn ecological model one ammensal and two mutually helping species, to study the stability analysis with distributed type of delay. The co existing stated is obtained and studied the analysis of local stability at this point and observed that the system is asymptotically stable. By constructing the Lyapunov's function, the global stability is studied. Using the numerical simulation in support of stability analysis, the dynamics of the system is studied. We considered two numerical examples, with and without time delay arguments. The impact of time delay with different kernel values is studied and observed that the system is neutrally stable, so time delay plays a vital role in the dynamics of ecological systems. We observed that the delay arguments played a key role in increasing the populations of the two mutually helping species(y & z).

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