JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1948-1956
https://publishoa.com
ISSN: 1309-3452

Mersenne Representation Hybrinomials

B. MALINI DEVI and S. DEVIBALA

School of Mathematics,

Madurai Kamaraj University, Tamil Nadu 625 021.

Email: bmalini22.mdu@gmail.com, devibala27@yahoo.com

Received 2022 April 02**; Revised** 2022 May 20**; Accepted** 2022 June 18**.**

Abstract:

In this paper, we introduce the Mersenne representation hybrinomials, that is, polynomials which are a generalization of Mersenne hybrid numbers. Also, we present some properties of Mersenne hybrinomials. We obtain generating functions, exponential generating functions and Binet formulas for Mersenne and Mersenne - Lucas hybrinomials. Also, we verify some well-known identities for Mersenne and Mersenne - Lucas hybrinomials.

Keywords: Mersenne sequence, Mersenne-Lucas sequence, Hybrid numbers, Polynomials, Hybrinomials.

2020 Mathematics Subject Classification: 11B39, 11B99

Introduction

Some studies about Mersenne sequence have been published such as [5] and Mersenne-Lucas sequence were introduced in [7]. Many papers are dedicated to Mersenne sequence such as [3, 8, 10, 11]. More generalization can be found in [2]. A natural extension of Mersenne sequence is given by Mersenne polynomial and defined as follows.

For any variable quantity ξ, the Mersenne polynomial $M_n(\xi)$ is defined as $M_n(\xi) = 3\xi M_{n-1}(\xi) - 2M_{n-2}(\xi)$ for $n \geq 2$ with $M_0(\xi) = 0, M_1(\xi) = 1$.

The Mersenne-Lucas polynomial $ML_n(\xi)$ is defined as $ML_n(\xi) = 3\xi ML_{n-1}(\xi) - 2ML_{n-2}(\xi)$ for $n \geq 2$ with initial terms $ML_0(\xi) = 2, ML_1(\xi) = 3\xi$. For polynomial representation one can refer [1, 4].

Binet formulas for Mersenne and Mersenne-Lucas polynomials are of the form

$$M_n(\xi) = \frac{q_1^n(\xi) - q_2^n(\xi)}{q_1(\xi) - q_2(\xi)}$$

and

$$ML_n(\xi) = q_1^n(\xi) + q_2^n(\xi)$$

where $q_1(\xi) = \frac{3\xi+\sqrt{9\xi^2-8}}{2}$, $q_2(\xi) = \frac{3\xi-\sqrt{9\xi^2-8}}{2}$ are roots of the characteristic equation $x^2 - 3\xi x + 2 = 0$.

Ozdemir [9] defined hybrid numbers $\mathcal{H}$ as a composition of dual, complex, hyperbolic numbers satisfying the relation $ih = -hi = i + \varepsilon$ and it is of the form

$$\mathcal{H} = z_0 + z_1 i + z_2 \varepsilon + z_3 h,$$

where $z_0, z_1, z_2, z_3 \in \mathbb{R}$ and $i,\ \varepsilon,\ h$ are operators such that $i^2 = -1,\ \varepsilon^2 = 0,\ h^2 = 1$.

We can associate Mersenne and Mersenne-Lucas numbers with hybrid numbers, we get the nth Mersenne hybrid number $M\mathcal{H}_n$ and the nth Mersenne-Lucas hybrid number $ML\mathcal{H}_n$ are defined as

$M\mathcal{H}_n = M_n + iM_{n+1} + \varepsilon M_{n+2} + hM_{n+3}$

$ML\mathcal{H}_n = ML_n + iML_{n+1} + \varepsilon ML_{n+2} + hML_{n+3}$

where i, ε, h are hybrid units, see [12, 14].

In [6, 13] Liana et al introduced Pell and Pell-Lucas hybrinomials, Fibonacci and Lucas hybrinomials. The main aim of this paper is to introduce Mersenne hybrinomials.

For $n \geq 0$, Mersenne and Mersenne-Lucas hybrinomials are defined by

$MH_n(\xi) = M_n(\xi) + iM_{n+1}(\xi) + \varepsilon M_{n+2}(\xi) + hM_{n+3}(\xi)$

and

$MLH_n(\xi) = ML_n(\xi) + iML_{n+1}(\xi) + \varepsilon ML_{n+2}(\xi) + hML_{n+3}(\xi)$

where $M_n(\xi)$ is the nth Mersenne polynomial and $MLH_n(\xi)$ is the nth Mersenne-Lucas polynomial and i, ε, h are hybrid units.

For example, $MH_5(\xi) = M_5(\xi) + iM_6(\xi) + \varepsilon M_7(\xi) + hM_8(\xi)$,

where $M_5(\xi) = 81\xi^3 - 54\xi^2 + 4$,

$M_6(\xi) = 243\xi^4 - 216\xi^3 + 36\xi$,

$M_7(\xi) = 729\xi^5 - 648\xi^4 - 162\xi^3 + 216\xi^2 - 8$,

$M_8(\xi) = 2187\xi^6 - 1944\xi^5 - 972\xi^4 + 1080\xi^3 - 96\xi$.

Also, $MLH_4(\xi) = ML_4(\xi) + iML_5(\xi) + \varepsilon ML_6(\xi) + hML_7(\xi)$,

where $ML_4(\xi) = 81\xi^4 - 72\xi^2 + 8$,

$ML_5(\xi) = 243\xi^5 - 270\xi^3 + 60\xi$,

$ML_6(\xi) = 729\xi^6 - 972\xi^4 + 324\xi^2 - 16$,

$ML_7(\xi) = 2187\xi^7 - 3402\xi^5 + 1512\xi^3 - 168\xi$.

For $\xi = 1$, we obtain Mersenne hybrid numbers and Mersenne-Lucas hybrid numbers.

Theorem 1 For any variable quantity ξ, we have

$MH_n(\xi) = 3\xi MH_{n-1}(\xi) - 2MH_{n-2}(\xi)$ for $n \geq 2$

with $MH_0(\xi) = i + \varepsilon(3\xi) + h(9\xi^2 - 2)$ and $MH_1(\xi) = 1 + i(3\xi) + \varepsilon(9\xi^2 - 2) + h(27\xi^3 - 12\xi)$.

Proof

If $n = 2$ we have

$MH_2(\xi) = 3\xi MH_1(\xi) - 2MH_0(\xi)$

$= 3\xi[1 + i(3\xi) + \varepsilon(9\xi^2 - 2) + h(27\xi^3 - 12\xi)] - 2[i + \varepsilon(3\xi) + h(9\xi^2 - 2)]$

$= 3\xi + i(9\xi^2 - 2) + \varepsilon(27\xi^3 - 12\xi) + h(81\xi^4 - 54\xi^2 + 4)$

$= M_2(\xi) + iM_3(\xi) + \varepsilon M_4(\xi) + hM_5(\xi)$

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1948-1956
https://publishoa.com
ISSN: 1309-3452

By using the definition of Mersenne polynomial we have

$MH_n(\xi) = M_n(\xi) + iM_{n+1}(\xi) + \varepsilon M_{n+2}(\xi) + hM_{n+3}(\xi)$

$= [3\xi M_{n-1}(\xi) - 2M_{n-2}(\xi)] + i[3\xi M_n(\xi) - 2M_{n-1}(\xi)] + \varepsilon[3\xi M_{n+1}(\xi) - 2M_n(\xi)] + h[3\xi M_{n+2}(\xi) - 2M_{n+1}(\xi)]$

$= 3\xi MH_{n-1}(\xi) - 2MH_{n-2}(\xi)$

Theorem 2 For any arbitrary quantity ξ, we have

$MLH_n(\xi) = 3\xi MLH_{n-1}(\xi) - 2MLH_{n-2}(\xi)$ for $n \geq 2$

with $MLH_0(\xi) = 2 + 3\xi i + \varepsilon(9\xi^2 - 4) + h(27\xi^3 - 18\xi)$ and $MLH_1(\xi) = 3\xi + i(9\xi^2 - 4) + \varepsilon(27\xi^3 - 18\xi) + h(81\xi^4 - 72\xi^2 + 8)$.

Proof

If $n = 2$ we have

$MLH_2(\xi) = 3\xi MLH_1(\xi) - 2MLH_0(\xi)$

$= 3\xi[3\xi + i(9\xi^2 - 4) + \varepsilon(27\xi^3 - 18\xi) + h(81\xi^4 - 72\xi^2 + 8)] - 2[2 + 3\xi i + \varepsilon(9\xi^2 - 4) + h(27\xi^3 - 18\xi)]$

$= (9\xi^2 - 4) + i(27\xi^3 - 18\xi) + \varepsilon(81\xi^4 - 72\xi^2 + 8) + h(243\xi^4 - 270\xi^3 + 60\xi)$

$= ML_2(\xi) + iML_3(\xi) + \varepsilon ML_4(\xi) + hML_5(\xi)$

By using the definition of Mersenne-Lucas polynomial we have

$MLH_n(\xi) = ML_n(\xi) + iML_{n+1}(\xi) + \varepsilon ML_{n+2}(\xi) + hML_{n+3}(\xi)$

$= [3\xi ML_{n-1}(\xi) - 2ML_{n-2}(\xi)] + i[3\xi ML_n(\xi) - 2ML_{n-1}(\xi)] + \varepsilon[3\xi ML_{n+1}(\xi) - 2ML_n(\xi)] + h[3\xi ML_{n+2}(\xi) - 2ML_{n+1}(\xi)]$

$= 3\xi MLH_{n-1}(\xi) - 2MLH_{n-2}(\xi)$.

Theorem 3 The generating function for Mersenne hybrinomial sequence $\{MH_n(\xi)\}$ is

$$G(t) = \frac{MH_0(\xi) + [MH_1(\xi) - 3\xi MH_0(\xi)]t}{1 - 3\xi t + 2t^2}$$

where $MH_0(\xi) = i + \varepsilon(3\xi) + h(9\xi^2 - 2)$ and $MH_1(\xi) = 1 + i(3\xi) + \varepsilon(9\xi^2 - 2) + h(27\xi^3 - 12\xi)$

Proof

Let $G(t) = \sum_{n=0}^{\infty} MH_n(\xi)t^n$ be the generating function of the Mersenne hybrinomial sequence. Then

$G(t) = MH_0(\xi) + MH_1(\xi)t + MH_2(\xi)t^2 + MH_3(\xi)t^3 + \cdots$

Multiplying by $-3\xi t$ and $2t^2$, we get

$-3\xi G(t)t = -3\xi MH_0(\xi)t - 3\xi MH_1(\xi)t^2 - 3\xi MH_2(\xi)t^3 - \cdots$

$2G(t)t^2 = 2MH_0(\xi)t^2 + 2MH_1(\xi)t^3 + 2MH_2(\xi)t^4 + \cdots$

By adding these above three equalities, we get

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1948-1956
https://publishoa.com
ISSN: 1309-3452

$G(t)(1-3\xi t+2t^2)=MH_0(\xi)+[MH_1(\xi)-3\xi MH_0(\xi)]t$

$$G(t)=\frac{MH_0(\xi)+[MH_1(\xi)-3\xi MH_0(\xi)]t}{1-3\xi t+2t^2}$$

Theorem 4 Generating function for Mersenne-Lucas hybrinomial sequence $\{MLH_n(\xi)\}$ is

$$F(t)=\frac{MLH_0(\xi)+[MLH_1(\xi)-3\xi MLH_0(\xi)]t}{1-3\xi t+2t^2}$$

where $MLH_0(\xi)=2+3\xi i+\varepsilon(9\xi^2-4)+h(27\xi^3-18\xi)$ and $MLH_1(\xi)=3\xi+i(9\xi^2-4)+\varepsilon(27\xi^3-18\xi)+h(81\xi^4-72\xi^2+8)$.

Proof

Let $F(t)=\sum_{n=0}^{\infty}MLH_n(\xi)t^n$

$F(t)=MLH_0(\xi)+MLH_1(\xi)t+MLH_2(\xi)t^2+MLH_3(\xi)t^3+\cdots$

Multiplying by $-3\xi t$ and $2t^2$, we get

$-3\xi F(t)t=-3\xi MLH_0(\xi)t-3\xi MLH_1(\xi)t^2-3\xi MLH_2(\xi)t^3-\cdots$

$2F(t)t^2=2MLH_0(\xi)t^2+2MLH_1(\xi)t^3+2MLH_2(\xi)t^4+\cdots$

By adding these above three equalities, we get

$F(t)(1-3\xi t+2t^2)=MLH_0(\xi)+[MLH_1(\xi)-3\xi MLH_0(\xi)]t$

$$F(t)=\frac{MLH_0(\xi)+[MLH_1(\xi)-3\xi MLH_0(\xi)]t}{1-3\xi t+2t^2}$$

Theorem 5 (Binet formula)

Let $n\geq 0$ be any integer. Then

$$MH_n(\xi)=\frac{q_1^n(\xi)}{q_1(\xi)-q_2(\xi)}[1+iq_1(\xi)+\epsilon q_1^2(\xi)+hq_1^3(\xi)]-\frac{q_2^n(\xi)}{q_1(\xi)-q(\xi)}[1+iq_2(\xi)+\epsilon q_2^2(\xi)+hq_2^3(\xi)]$$

where $q_1(\xi)=\frac{3\xi+\sqrt{9\xi^2-8}}{2}$ and $q_2(\xi)=\frac{3\xi-\sqrt{9\xi^2-8}}{2}$.

Proof

$$MH_n(\xi)=M_n(\xi)+iM_{n+1}(\xi)+\varepsilon M_{n+2}(\xi)+hM_{n+3}(\xi)$$

$$=\frac{q_1^n(\xi)-q_2^n(\xi)}{q_1(\xi)-q_2(\xi)}+i\frac{q_1^{n+1}(\xi)-q_2^{n+1}(\xi)}{q_1(\xi)-q_2(\xi)}+\epsilon\frac{q_1^{n+2}(\xi)-q_2^{n+2}(\xi)}{q_1(\xi)-q_2(\xi)}+h\frac{q_1^{n+3}(\xi)-q_2^{n+3}(\xi)}{q_1(\xi)-q_2(\xi)}$$

$$=\frac{q_1^n(\xi)}{q_1(\xi)-q_2(\xi)}[1+iq_1(\xi)+\epsilon q_1^2(\xi)+hq_1^3(\xi)]-\frac{q_2^n(\xi)}{q_1(\xi)-q_2(\xi)}[1+iq_2(\xi)+\epsilon q_2^2(\xi)+hq_2^3(\xi)]$$

Theorem 6 (Binet formula)

Let $n\geq 0$ be any integer. Then

$$MLH_n(\xi)=q_1^n(\xi)[1+iq_1(\xi)+\epsilon q_1^2(\xi)+hq_1^3(\xi)]+q_2^n(\xi)[1+iq_2(\xi)+\epsilon q_2^2(\xi)+hq_2^3(\xi)]$$

where $q_1(\xi)=\frac{3\xi+\sqrt{9\xi^2-8}}{2}$ and $q_2(\xi)=\frac{3\xi-\sqrt{9\xi^2-8}}{2}$.

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1948-1956
https://publishoa.com
ISSN: 1309-3452

Proof

$MLH_n(\xi) = ML_n(\xi) + iML_{n+1}(\xi) + \varepsilon ML_{n+2}(\xi) + hML_{n+3}(\xi)$

$= [q_1^n(\xi) + q_2^n(\xi)] + i[q_1^{n+1}(\xi) + q_2^{n+1}(\xi)] + \epsilon[q_1^{n+2}(\xi) + q_2^{n+2}(\xi)] + h[q_1^{n+3}(\xi) + q_2^{n+3}(\xi)]$

$= q_1^n(\xi)[1 + iq_1(\xi) + \epsilon q_1^2(\xi) + hq_1^3(\xi)] + q_2^n(\xi)[1 + iq_2(\xi) + \epsilon q_2^2(\xi) + hq_2^3(\xi)].$

For simplicity of notation let

$\mathfrak{Q}(\xi) = q_1(\xi) - q_2(\xi)$

$\mathfrak{Q}^2(\xi) = 9\xi^2 - 8$

$\wedge_1(\xi) = 1 + iq_1(\xi) + \epsilon q_1^2(\xi) + hq_1^3(\xi)$

$\wedge_2(\xi) = 1 + iq_2(\xi) + \epsilon q_2^2(\xi) + hq_2^3(\xi)$

$q_1(\xi)q_2(\xi) = 2$

We can write Binet formulas for Mersenne hybrinomial and Mersenne-Lucas hybrinomial sequences as

$$MH_n(\xi) = \frac{q_1^n(\xi)}{\mathfrak{Q}(\xi)}\wedge_1(\xi) - \frac{q_2^n(\xi)}{\mathfrak{Q}(\xi)}\wedge_2(\xi)$$

$$MLH_n(\xi) = q_1^n(\xi)\wedge_1(\xi) + q_2^n(\xi)\wedge_2(\xi)$$

Theorem 7 For $n \geq 0$, exponential generating functions for Mersenne hybrinomial and Mersenne-Lucas hybrinomial sequences are

$$\sum_{n=0}^{\infty} MH_n(\xi)\frac{t^n}{n!} = \frac{\wedge_1(\xi)}{\mathfrak{Q}(\xi)}e^{q_1(\xi)t} - \frac{\wedge_2(\xi)}{\mathfrak{Q}(\xi)}e^{q_2(\xi)t}$$

$$\sum_{n=0}^{\infty} MLH_n(\xi)\frac{t^n}{n!} = \wedge_1(\xi)e^{q_1(\xi)t} + \wedge_2(\xi)e^{q_2(\xi)t}$$

Proof

$$\sum_{n=0}^{\infty} MH_n(\xi)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left[\frac{q_1^n(\xi)\wedge_1(\xi) - q_2^n(\xi)\wedge_2(\xi)}{\mathfrak{Q}(\xi)}\right]\frac{t^n}{n!}$$

$$= \frac{\wedge_1(\xi)}{\mathfrak{Q}(\xi)}\sum_{n=0}^{\infty}\frac{q_1^n(\xi)t^n}{n!} - \frac{\wedge_2(\xi)}{\mathfrak{Q}(\xi)}\sum_{n=0}^{\infty}\frac{q_2^n(\xi)t^n}{n!}$$

$$= \frac{\wedge_1(\xi)}{\mathfrak{Q}(\xi)}e^{q_1(\xi)t} - \frac{\wedge_2(\xi)}{\mathfrak{Q}(\xi)}e^{q_2(\xi)t}$$

$$\sum_{n=0}^{\infty} MLH_n(\xi)\frac{t^n}{n!} = \sum_{n=0}^{\infty}[q_1^n(\xi)\wedge_2(\xi) + q_2^n(\xi)\wedge_2(\xi)]\frac{t^n}{n!}$$

$$= \wedge_1(\xi)\sum_{n=0}^{\infty}\frac{q_1^n(\xi)t^n}{n!} - \wedge_2(\xi)\sum_{n=0}^{\infty}\frac{q_2^n(\xi)t^n}{n!}$$

$$= \wedge_1(\xi)e^{q_1(\xi)t} + \wedge_2(\xi)e^{q_2(\xi)t}$$

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1948-1956
https://publishoa.com
ISSN: 1309-3452

Theorem 8

Let $n \geq 0,\ s \geq 0$ be integers such that $n \geq s$, then

$$MH_{n-s}(\xi)MH_{n+s}(\xi) - [MH_n(\xi)]^2 = \frac{2^n}{\mathfrak{Q}^2(\xi)}\left[\wedge_2(\xi)\ \wedge_1(\xi)\ \left(1 - \frac{q_1^s(\xi)}{q_2^s(\xi)}\right) + \wedge_1(\xi)\ \wedge_2(\xi)\left(1 - \frac{q_2^s(\xi)}{q_1^s(\xi)}\right)\right]$$

and

$$MLH_{n-s}(\xi)MLH_{n+s}(\xi) - [MLH_n(\xi)]^2 = 2^n\left[\wedge_2(\xi)\ \wedge_1(\xi)\left(\frac{q_1^s(\xi)}{q_2^s(\xi)} - 1\right) + \wedge_1(\xi)\ \wedge_2(\xi)\left(\frac{q_2^s(\xi)}{q_1^s(\xi)} - 1\right)\right]$$

Proof

$$MH_{n-s}(\xi)MH_{n+s}(\xi) - [MH_n(\xi)]^2$$

$$= \frac{q_2^n(\xi)q_1^n(\xi)\wedge_2(\xi)\ \wedge_1(\xi)}{\mathfrak{Q}^2(\xi)}\left(1 - \frac{q_1^s(\xi)}{q_2^s(\xi)}\right) + \frac{q_1^n(\xi)q_2^n(\xi)\wedge_1(\xi)\wedge_2(\xi)}{\mathfrak{Q}^2(\xi)}\left(1 - \frac{q_2^s(\xi)}{q_1^s(\xi)}\right)$$

$$= \frac{2^n}{\mathfrak{Q}^2(\xi)}\left[\wedge_2(\xi)\ \wedge_1(\xi)\ \left(1 - \frac{q_1^s(\xi)}{q_2^s(\xi)}\right) + \wedge_1(\xi)\wedge_2(\xi)\left(1 - \frac{q_2^s(\xi)}{q_1^s(\xi)}\right)\right]$$

$$MLH_{n-s}(\xi)MLH_{n+s}(\xi) - [MLH_n(\xi)]^2$$

$$= (q_1^{n-s}(\xi)\wedge_1(\xi) + q_2^{n-s}(\xi)\wedge_2(\xi))(q_1^{n+s}(\xi)\wedge_1(\xi) + q_2^{n+s}(\xi)\wedge_2(\xi)) - (q_1^n(\xi)\wedge_1(\xi) + q_2^n(\xi)\wedge_2(\xi))(q_1^n(\xi)\wedge_1(\xi) + q_2^n(\xi)\wedge_2(\xi))$$

$$= q_2^n(\xi)q_1^n(\xi)\wedge_2(\xi)\ \wedge_1(\xi)\left(\frac{q_1^s(\xi)}{q_2^s(\xi)} - 1\right) + q_1^n(\xi)q_2^n(\xi)\wedge_1(\xi)\ \wedge_2(\xi)\left(\frac{q_2^s(\xi)}{q_1^s(\xi)} - 1\right)$$

$$= 2^n\left[\wedge_2(\xi)\ \wedge_1(\xi)\left(\frac{q_1^s(\xi)}{q_2^s(\xi)} - 1\right) + \wedge_1(\xi)\ \wedge_2(\xi)\left(\frac{q_2^s(\xi)}{q_1^s(\xi)} - 1\right)\right]$$

For $s = 1$, we obtain Cassini identities for Mersenne hybrinomials and Mersenne-Lucas hybrinomials.

Theorem 9

For n to be any positive integer, then

$$MH_{n-1}(\xi)MH_{n+1}(\xi) - [MH_n(\xi)]^2 = \frac{2^{n-1}}{\mathfrak{Q}(\xi)}[\wedge_1(\xi)\ \wedge_2(\xi)\ q_2(\xi) - \wedge_2(\xi)\ \wedge_1(\xi)q_1(\xi)]$$

and

$$MLH_{n-1}(\xi)MLH_{n+1}(\xi) - [MLH_n(\xi)]^2 = 2^{n-1}[\wedge_2(\xi)\ \wedge_1(\xi)q_1(\xi) - \wedge_1(\xi)\ \wedge_2(\xi)q_2(\xi)]$$

Theorem 10 Let $n \geq 0, m \geq 0$ be integers such that $m \geq n$. Then

$$MH_m(\xi)MH_{n+1}(\xi) - MH_{m+1}(\xi)MH_n(\xi) = \frac{2^n}{\mathfrak{Q}(\xi)}[q_1^{m-n}(\xi)\wedge_1(\xi)\ q_2^*(\xi) - q_2^{m-n}(\xi)\wedge_2(\xi)\ \wedge_2(\xi)]$$

and

$$MLH_m(\xi)MLH_{n+1}(\xi) - MLH_{m+1}(\xi)MLH_n(\xi) = 2^n\mathfrak{Q}(\xi)[q_2^{m-n}(\xi)\wedge_2(\xi)\ \wedge_1(\xi) - q_1^{m-n}(\xi)\wedge_2(\xi)\ \wedge_1(\xi)]$$

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1948-1956
https://publishoa.com
ISSN: 1309-3452

Proof

$MH_m(\xi)MH_{n+1}(\xi) - MH_{m+1}(\xi)MH_n(\xi)$

$= \frac{q_2^m(\xi)q_1^n(\xi) \wedge_2(\xi) \wedge_1(\xi)}{\mathfrak{Q}^2(\xi)}\big(q_2(\xi) - q_1(\xi)\big) + \frac{q_1^m(\xi)q_2^n(\xi) \wedge_1(\xi) \wedge_2(\xi)}{\mathfrak{Q}^2(\xi)}\big(q_1(\xi) - q_2(\xi)\big)$

$= \frac{[q_1^n(\xi)q_2^n(\xi)]q_1^{m-n}(\xi) \wedge_1(\xi) \wedge_2(\xi)}{\mathfrak{Q}(\xi)} - \frac{[q_2^n(\xi)q_1^n(\xi)]q_2^{m-n}(\xi) \wedge_1(\xi) \wedge_2(\xi)}{\mathfrak{Q}(\xi)}$

$= \frac{2^n}{\mathfrak{Q}(\xi)}[q_1^{m-n}(\xi) \wedge_1(\xi) \wedge_2(\xi) - q_2^{m-n}(\xi) \wedge_2(\xi) \wedge_1(\xi)]$

$MLH_m(\xi)MLH_{n+1}(\xi) - MLH_{m+1}(\xi)MLH_n(\xi)$

$= q_2^m(\xi)q_1^n(\xi) \wedge_2(\xi) \wedge_1(\xi)\big(q_1(\xi) - q_2(\xi)\big) + q_1^m(\xi)q_2^n(\xi) \wedge_1(\xi) \wedge_2(\xi)\big(q_2(\xi) - q_1(\xi)\big)$

$= \mathfrak{Q}(\xi)\{[q_2^n(\xi)q_1^n(\xi)]q_2^{m-n}(\xi) \wedge_1(\xi) \wedge_2(\xi) - [q_1^n(\xi)q_2^n(\xi)]q_1^{m-n}(\xi) \wedge_1(\xi) \wedge_2(\xi)\}$

$= 2^n\mathfrak{Q}(\xi)[q_2^{m-n}(\xi) \wedge_2(\xi) \wedge_1(\xi) - q_1^{m-n}(\xi) \wedge_1(\xi) \wedge_2(\xi)]$

Theorem 11 Let $n \geq 0, m \geq 0$ be integers then

i. $MH_m(\xi)MLH_n(\xi) - MLH_m(\xi)MH_n(\xi)$
$= \frac{2}{\mathfrak{Q}(\xi)}[q_1^m(\xi)q_2^n(\xi) \wedge_1(\xi) \wedge_2(\xi) - q_2^m(\xi)q_1^n(\xi) \wedge_2(\xi) \wedge_1(\xi)]$

ii. $MH_m(\xi)MLH_n(\xi) + MLH_m(\xi)MH_n(\xi) = \frac{2}{\mathfrak{Q}(\xi)}[q_1^{m+n}(\xi) \wedge_1(\xi)^2 - q_2^{m+n}(\xi) \wedge_2(\xi)^2]$

Proof

i. $MH_m(\xi)MLH_n(\xi) - MLH_m(\xi)MH_n(\xi)$

$= \left(\frac{q_1^m(\xi)}{\mathfrak{Q}(\xi)} \wedge_1(\xi) - \frac{q_2^m(\xi)}{\mathfrak{Q}(\xi)} \wedge_2(\xi)\right)(q_1^n(\xi) \wedge_1(\xi) + q_2^n(\xi) \wedge_2(\xi)) - (q_1^m(\xi) \wedge_1(\xi) +$
$q_2^m(\xi) \wedge_2(\xi))\left(\frac{q_1^n(\xi)}{Q(\xi)} \wedge_1(\xi) - \frac{q_2^n(\xi)}{Q(\xi)} \wedge_2(\xi)\right)$

$= \frac{2q_1^m(\xi)q_2^n(\xi)\wedge_1(\xi) \wedge_2(\xi)}{\mathfrak{Q}(\xi)} - \frac{2q_2^m(\xi)q_1^n(\xi)\wedge_2(\xi) \wedge_2(\xi)}{\mathfrak{Q}(\xi)}$

$= \frac{2}{\mathfrak{Q}(\xi)}[q_1^m(\xi)q_2^n(\xi) \wedge_1(\xi) \wedge_2(\xi) - q_2^m(\xi)q_1^n(\xi) \wedge_2(\xi) \wedge_1(\xi)]$

ii. $MH_m(\xi)MLH_n(\xi) + MLH_m(\xi)MH_n(\xi)$

$= \left(\frac{q_1^m(\xi)}{\mathfrak{Q}(\xi)} \wedge_1(\xi) - \frac{q_2^m(\xi)}{\mathfrak{Q}(\xi)} \wedge_2(\xi)\right)(q_1^n(\xi) \wedge_1(\xi) + q_2^n(\xi) \wedge_2(\xi)) + (q_1^m(\xi) \wedge_1(\xi) +$
$q_2^m(\xi) \wedge_2(\xi))\left(\frac{q_1^n(\xi)}{\mathfrak{Q}(\xi)} \wedge_1(\xi) - \frac{q_2^n(\xi)}{\mathfrak{Q}(\xi)} \wedge_2(\xi)\right)$

$= 2\frac{q_1^{m+n}(\xi)\wedge_1(\xi)^2}{\mathfrak{Q}(\xi)} - 2\frac{q_2^{m+n}(\xi)\wedge_2(\xi)^2}{\mathfrak{Q}(\xi)}$

$= \frac{2}{\mathfrak{Q}(\xi)}[q_1^{m+n}(\xi) \wedge_1(\xi)^2 - q_2^{m+n}(\xi) \wedge_2(\xi)^2]$

Theorem 12 Let $n \geq 0, r \geq 0,\ s \geq 0$ be integers then

$$MH_{n+r}(\xi)MLH_{n+s}(\xi)-MH_{n+s}(\xi)MLH_{n+r}(\xi) = \frac{2^n \wedge_1(\xi)\ \wedge_2(\xi)}{\mathfrak{Q}(\xi)}[q_1^r(\xi)q_2^s(\xi)-q_1^s(\xi)q_2^r(\xi)] - \frac{2^n \wedge_2(\xi)\ \wedge_1(\xi)}{\mathfrak{Q}(\xi)}[q_2^r(\xi)q_1^s(\xi)-q_2^s(\xi)q_1^r(\xi)]$$

Proof

$$MH_{n+r}(\xi)MLH_{n+s}(\xi)-MH_{n+s}(\xi)MLH_{n+r}(\xi)$$

$$=\left(\frac{q_1^{n+r}(\xi)}{\mathfrak{Q}(\xi)}\wedge_1(\xi)-\frac{q_2^{n+r}(\xi)}{\mathfrak{Q}(\xi)}\wedge_2(\xi)\right)(q_1^{n+s}(\xi)\wedge_1(\xi)+q_2^{n+s}(\xi)\wedge_2(\xi)) - \left(\frac{q_1^{n+s}(\xi)}{\mathfrak{Q}(\xi)}\wedge_1(\xi)-\frac{q_2^{n+s}(\xi)}{\mathfrak{Q}(\xi)}\wedge_2(\xi)\right)(q_1^{n+r}(\xi)\wedge_1(\xi)+q_2^{n+r}(\xi)\wedge_2(\xi))$$

$$=\frac{q_1^n(\xi)q_2^n(\xi)\wedge_1(\xi)\ \wedge_2(\xi)}{\mathfrak{Q}(\xi)}[q_1^r(\xi)q_2^s(\xi)-q_1^s(\xi)q_2^r(\xi)]-\frac{q_2^n(\xi)q_1^n(\xi)\wedge_2(\xi)\ \wedge_1(\xi)}{\mathfrak{Q}(\xi)}[q_2^r(\xi)q_1^s(\xi)-q_2^s(\xi)q_1^r(\xi)]$$

$$=\frac{2^n \wedge_1(\xi)\ \wedge_2(\xi)}{\mathfrak{Q}(\xi)}[q_1^r(\xi)q_2^s(\xi)-q_1^s(\xi)q_2^r(\xi)]-\frac{2^n \wedge_2(\xi)\ \wedge_1(\xi)}{\mathfrak{Q}(\xi)}[q_2^r(\xi)q_1^s(\xi)-q_2^s(\xi)q_1^r(\xi)]$$

Theorem 13 Let $n \geq 0$ then

$$\mathfrak{Q}^2(\xi)[MH_n(\xi)]^2-[MLH_n(\xi)]^2 = -2^{n+1}[\wedge_1(\xi)\ \wedge_2(\xi)+\wedge_2(\xi)\wedge_2(\xi)]$$

Proof

$$\mathfrak{Q}^2(\xi)[MH_n(\xi)]^2-[MLH_n(\xi)]^2$$

$$=\mathfrak{Q}^2(\xi)\left[\frac{q_1^n(\xi)}{\mathfrak{Q}(\xi)}\wedge_1(\xi)-\frac{q_2^n(\xi)}{\mathfrak{Q}(\xi)}\wedge_2(\xi)\right]^2-[q_1^n(\xi)\wedge_1(\xi)+q_2^n(\xi)\wedge_2(\xi)]^2$$

$$=-2q_2^n(\xi)q_1^n(\xi)\wedge_2(\xi)\wedge_1(\xi)-2q_1^n(\xi)q_2^n(\xi)\wedge_1(\xi)\ \wedge_2(\xi)$$

$$=-2^{n+1}[\wedge_1(\xi)\ \wedge_2(\xi)+\wedge_2(\xi)\ \wedge_1(\xi)]$$

Conclusion

In this communication, we introduced and studied polynomials which are a generalization of Mersenne hybrid numbers and so called Mersenne hybrinomials. We associated Lucas numbers with Mersenne hybrinomials. In the same angle, one may search for Mersenne hybrinomials with hypercomplex numbers and so on.

References

1. Ali Boussayoud, Mohamed Kerada, Nesrine Harrouche, On the k-Lucas numbers and Lucas polynomials, Turkish Journal of Analysis and Number Theory, Vol. 5, No. 4, pp. 121-125, 2017. doi: 10.12691/tjant-5-4-1
2. Ali Boussayoud, Mourad Chelgham and Souhila Boughaba, On some identities and generating functions for Mersenne numbers and polynomials, Turkish Journal of Analysis and Number Theory, Vol. 6, No. 3, pp. 93-97, 2018.doi: 10.12691/tjant-6-3-5
3. Amelia Carolina Sparavigna, On a generalized sum of the Mersenne numbers, Amelia Carolina Sparavigna, hal-01796401, 2018. doi: 10.5281/zenodo.1250048
4. Horzum. T and Kocer. E.G, On some properties of Horadam polynomials, Int. Math. Forum, Vol. 4, No. 25, pp. 1243-1252, 2009.
5. Koshy. T, Gao. Z, Catalan numbers with Mersenne subscripts, Math. Scientist, Vol. 38, No. 2, pp. 86-91, 2013.
6. Liana. M, Szynal-Liana. A and Wloch. I, On Pell hybrinomials, Miskolc Math. Notes, Vol. 20, No. 2, pp. 1051-1062, 2019. doi: 10.18514/MMN.2019.2971

7. Malini Devi. B, Devibala. S, Some Sums Formulae for Products of terms of Mersenne and Mersenne-Lucas Numbers, Adalya Journal, Vol. 10, Issue 8, pp. 74-79, 2021. doi: https://doi.org/10.37896/aj9.8/008
8. Malini Devi. B, Devibala. S, A View on the Mersenne and Mersenne-Lucas Numbers, GIS Science Journal, Vol. 8, Issue 8, pp. 784-789, 2021. doi: 20.18001.GSJ.2021.V8I8.21.37829
9. Ozdemir. M, Introduction to Hybrid Numbers, Adv. Appl. Clifford Algebras, Vol. 28, No. 11, 2018. doi: 10.1007/s00006-018-0833-3
10. Paula Catarino, Helena Campos and Paulo Vasco, On the Mersenne sequences, Annales Mathematicae et informaticae, Vol. 46, pp. 37-53, 2016.
11. Robert Granger and Andrew Moss, Generalised Mersenne number revisited, Mathematics of computation, Vol. 82, No. 284, pp. 2389-2420, 2013. doi: https://doi.org/10.1090/S0025-5718-2013-02704-4
12. Szynal-Liana. A and Wloch.I, On Jacobsthal and Jacobsthal-Lucas Hybrid Numbers, In Annales Mathematicae Silesianae, Vol. 33, No. 1, pp. 276-283, 2019. doi: https://doi.org/10.2478/amsil-2018-0009
13. Szynal-Liana. A and Wloch.I, Introduction to Fibonacci and Lucas hybrinomials, Complex Var. Elliptic Equ., Vol. 65, No. 10, pp. 1736-1747, 2020. https://doi.org/10.1080/17476933.2019.1681416
14. Szynal-Liana. A, The Horadam hybrid numbers, Discuss. Math. Gen. Algebra Appl., Vol. 38, No. 1, pp. 91-98, 2018. doi: 10.7151/dmgaa.1287