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Algorithms for finding a Neighbourhood Total Restrained Dominating Set of Interval Graphs and Circular-Arc Graphs

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Abstract

A set $S_{ntr} \subset V(G)$ is a neighbourhood total restrained dominating set of a graph G if all the vertices in $V - S_{ntr}$ has one or more adjacent vertex in S_{ntr} as well as in $V - S_{ntr}$ and also the each vertex in S_{ntr} is adjacent to at least a vertex in S_{ntr} and if the induced sub-graph $\langle N(S_{ntr})\rangle$ has no isolated vertex. The cardinality of a minimum total restrained dominating set S_{ntr} is called total restrained dominating number and is represented as γ_{ntr}.

In this paper, we are introducing Neighborhood total restrained domination and develop an algorithm to find neighbourhood total restrained dominating set for Interval graphs and circular-arc graphs.

Keywords: Dominating set, Neighbourhood, total dominating set, restrained dominating set, total restrained dominating set.

1. Introduction

A set $S \subset V(G)$ is a dominating set of a graph G, if each vertex in $V \setminus \{S\}$ is adjacent to one or more vertex in S. A dominating set S is a restrained dominating set if each vertex in $V \setminus \{S\}$ has an adjacent vertex in $V \setminus \{S\}$ [1]. The idea of restrained domination was introduced by Telle[2]. The set S is said to be a total restrained dominating set of a graph G, if every vertex in S has at least a neighbour in S [5][3]. And the induced sub-graph $\langle N(S)\rangle$ has no isolated vertex then S is called a neighbourhood total restrained dominating set of graph G. The cardinality of a minimum total restrained dominating set S_{ntr} is called total restrained dominating number and is represented as γ_{ntr}. Henning[4] proved $\gamma_{ntr} \le n - \frac{\Delta}{2} - 1$ for a graph with $n \ge 4$ and maximum degree $\Delta \le n-2$. Various critical concepts has been studied to investigate the removal and addition of an edge on restrained domination number in signed graphs[6]. Resolving restrained dominating set has been characterized for lexicographic product of graphs[7].

In this paper we are introducing the definition for Neighoborhood total restrained dominating set and algorithm to find a NTRD -set for interval family, Circular-Arc family and also few results. For algorithms we are using the interval graphs of degree at most 5 or 6 and the circular arc family which has no path of size 3 or more than 3, because the algorithm fails for such circular-arc family of graphs. If a circular-arc graph has a path of size 2 with a pendent vertex at the end vertex, is not considerable.

Notations:

- NTRD - Neighborhood Total Restrained Dominating Set
- γ_{ntr} - Neighborhood Total Restrained Number
- NI - first non intersecting vertex
- nrd - neighborhood
- nrd^{+} - All succeeding neighbors
- nrd^{+1} - First succeeding neighbor
- nrd^{-1} - First preceding neighbor

2. Main Results

Observation 2.1: *For any path of order* n

$$\gamma_{ntr}(P_n) = \begin{cases} \frac{n}{2}+1 & n=6,10,14,... \\ \frac{n}{2}+2 & n=8,12,16,... \\ \frac{(n+1)}{2}+1 & n=7,11,15,... \\ \frac{(n+1)}{2}+2 & n=9,13,17,... \end{cases}$$

Proof: NTRD -set for a path of order $n \le 5$ does not exists, the dominating does not satisfy the condition of NTRD -set.

Case 1: If n is even

Case 1.1: If $n = 6, 10, 14, ...$

For $n = 6$, $\gamma_{ntr} = 4$

For $n = 10$, $\gamma_{ntr} = 6$

For $n = 14$, $\gamma_{ntr} = 8$

By proceeding the same, we may generalize it into

$$\gamma_{ntr}(P_n) = \frac{n}{2} + 1; \quad n = 6, 10, 14, ...$$

Case 1.2: If $n = 8, 12, 16, ...$

For $n = 8$, $\gamma_{ntr} = 6$

For $n = 12$, $\gamma_{ntr} = 8$

For $n = 16$, $\gamma_{ntr} = 10$

In general we can write,

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$\gamma_{ntr}(P_n) = \frac{n}{2} + 2$; $n = 8, 12, 16, ...$

Similarly,

Case 2: If n is odd

Case 2.1: If $n = 7, 11, 15, ...$

For $n = 7$, $\gamma_{ntr} = 5$

For $n = 11$, $\gamma_{ntr} = 7$

For $n = 15$, $\gamma_{ntr} = 9$

In general we can write, $\gamma_{ntr}(P_n) = \frac{(n+2)}{2} + 1$

Case 2.2: If $n = 9, 13, 17, ...$

For $n = 9$, $\gamma_{ntr} = 7$

For $n = 13$, $\gamma_{ntr} = 9$

For $n = 16$, $\gamma_{ntr} = 11$

In general we can write, $\gamma_{ntr}(P_n) = \frac{(n+1)}{2} + 2$

Observation 2.2 *For any cycle*

$$\gamma_{ntr}(C_n) = \begin{cases} \frac{n}{2} & n=\{2m \backslash m \geq 2 \text{ is even}\} \\ \frac{n}{2}+1 & n=\{2m \backslash m \geq 3 \text{ is odd}\} \\ \frac{(n+1)}{2} & n=\{2m+1 \backslash m \geq 2 \text{ is even}\} \\ \frac{(n+3)}{2} & n=\{2m+1 \backslash m \geq 3 \text{ is odd}\} \end{cases}$$

Observation 2.3 *For any complete graph* $\gamma_{ntr}(K_n) = 2$

Observation 2.4 *For any wheel graph* $\gamma_{ntr}(W_n) = 2$

Observation 2.5 *For windmill graphs*

$$\gamma_{ntr}(W_n^m) = \begin{cases} \text{does not exists} & \text{for } n=3 \\ 2 & \text{for } n \geq 4 \end{cases}$$

Observation 2.6 *Neighborhood total restrained domination exists for a graph G of order* $n \geq 4$.

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Observation 2.7 *For any Graph* $\gamma_{ntr}(G) \geq 2$

Algorithm 1: Algorithm to find a neighborhood total restrained dominating set for an Interval family

Input: $I = \{i_1, i_2, i_3, \dots, i_n\}$

Output: minimum neighborhood total restrained dominating set for the given interval family

Step 1: NTRD = {}

Step 2: $x = i_1$

Step 3: $S = nrd[x]$

Step 4: If $S_1 = \{y \in S \setminus y$ is adjacent to all other intervals in $S\}$ then

Step 4.1: $a = max(S_1)$

Step 4.2: NTRD = NTRD $\cup$ $\{a\}$

Step 5: If there exists a pendent interval $i_p \in nrd(a)$ then

NTRD = NTRD $\cup$ $\{i_p\}$

else if there exists no pendent interval $i_p \in nrd(a)$ then

NTRD = NTRD $\cup$ $\{nrd^{+1}(a)\}$

else there exists no $nrd^{+}(a)$ then

NTRD = NTRD $\cup$ $\{nrd^{-1}(a)\}$

Step 6: $d = max(A)$

Step 7: If $x = NI(d) \neq \varnothing$ then

go to Step 2.

else

go to Step 8

Step 8: End

Illustration:

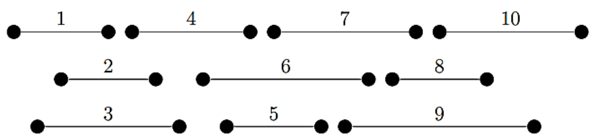

Figure 1: Interval Family

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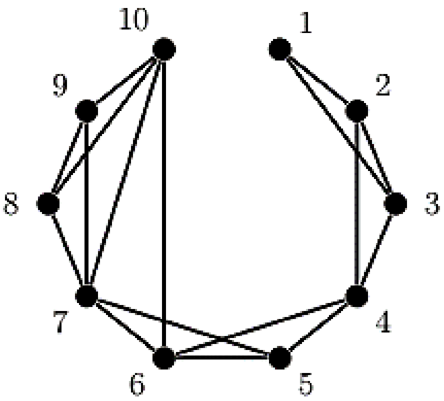

Figure 2: Interval Graph

Input: $I = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Step 1: NTRD = {}

Step 2: $x = 1$

Step 3: $S = nrd^{+}[x] = nrd^{+}[1] = \{1, 2, 3\}$

Step 4: S_1={1,2,3}

Step 4.1: a=3

Step 4.2: NTRD = NTRD$\cup\{a\} = \{\}\cup\{3\} = \{3\}$

Step 5: There exists no pendent vertex $i_p \in nrd(a)$ then

NTRD = NTRD $\cup$ $\{nrd^{+1}(a)\} = \{3\} \cup \{4\} = \{3, 4\}$

Step 6: $d = max(\{\text{NTRD}\}) = max(\{3, 4\}) = 4$

Step 7: $x = NI(d) = NI(4) = 7 \neq \emptyset$

Step 2: $x = 7$

Step 3: $S = nrd^{+}[x] = nrd^{+}[7] = \{7, 8, 9, 10\}$

Step 4: $S_1 = \{7, 8, 9, 10\}$

Step 4.1: $a = max(S_1) = 10$

Step 4.2: NTRD = NTRD$\cup\{10\} = \{3, 4\}\cup\{10\} = \{3, 4, 10\}$

Step 5: There exists no pendent vertex $i_p \in nrd(a)$ and also no $nrd^{+1}(a)$ then

NTRD = NTRD$\cup\{nrd^{-1}(10)\} = \{3, 4, 10\}\cup\{9\} = \{3, 4, 9, 10\}$

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Step 6: $d = max(\text{NTRD}) = max(\{3,4,9,10\} = 10$

Step 7: $x = NI(d) = NI(10) = \varnothing$

Step 8: End

Output: $\text{NTRD} = \{3,4,9,10\}$

Observation 2.8 *By using algorithm 1 the induced sub-graph of* NTRD *-set is always disconnected if* $n \geq 4 \in S$*, where* S *is* NTRD *-set.*

Theorem 2.1 *Let* i_p *be a pendent interval in an interval graph* G*, if an interval* $i \in nrd[i_p]$ *then the intervals* i *and* i_p *both must be in the neighbourhood total restrained dominating set* S_{ntr}.

Proof: Suppose, i and i_p are not in S, i.e., $i, i_p \not\subset S_{ntr}$

$i \not\subset S_{ntr}$ and $i_p \not\subset S_{ntr}$

Here, the following two cases will arise,

Case 1: If $i \in S_{ntr}$ and $i_p \not\subset S_{ntr}$

i is the only neighbour of i_p ($\because i_p$ is a pendent interval)

i_p is dominated by i

By the definition of NTRD, we have $\langle N(S_{ntr})\rangle$ has no isolated vertex(interval).

Here, $i_p \not\subset S_{ntr}$

$i_p \in \langle N(S_{ntr})\rangle$ is an isolated interval.

which is a contradiction

$\therefore i_p \in S_{ntr}$ and $i \in S_{ntr}$

Case 2: If $i \not\subset S_{ntr}$ and $i_p \in S_{ntr}$

i_p has no neighbourhood in S_{ntr} as $i \not\subset S_{ntr}$ is the only neighbourhood of i_p

which is again a contradiction

$\therefore i_p \in S_{ntr}$ and $i \in S_{ntr}$

From Case 1 and Case 2, both i and i_p must be in NTRD

Hence the proof.

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Algorithm 2: Algorithm to find NTRD -set for Circular-Arc graph

Input: Circular-Arc family

Output: A neighbourhood total restrained dominating set for a Circular-arc graph

Step 1: $\text{NTRD} = \{\}$

Step 2: $x = i_1$

Step 3: $S = nrd^{+}[x]$

Step 4: $S_1 = nrd^{-}[x]$

Step 5: $S_2 = \{y \,/\, y$ is adjacet to all other intervals in $S\}$

Step 5.1: $a = max(S_2)$

Step 5.2: $\text{NTRD} = \text{NTRD} \cup \{a\}$

Step 6: If $nrd(a)$ has a pendent vertex i_p then

$\text{NTRD} = \text{NTRD} \cup \{i_p\}$

else

$\text{NTRD} = \text{NTRD} \cup \{nrd^{+1}(a)\}$

Step 7: $d = max(\text{NTRD})$

Step 8: $x = NI(d) \neq \emptyset \not\subseteq nrd^{-}(1)$ then

go to Step 2

else

go to Step 9

Step 9: End

Illustration:

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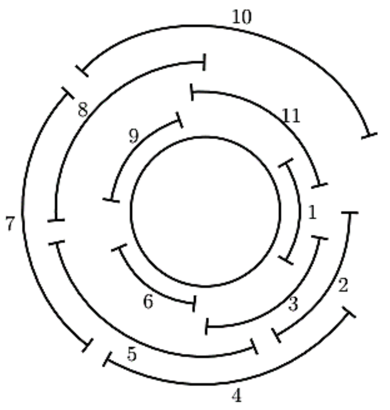

Figure 3: Circular – Arc Family

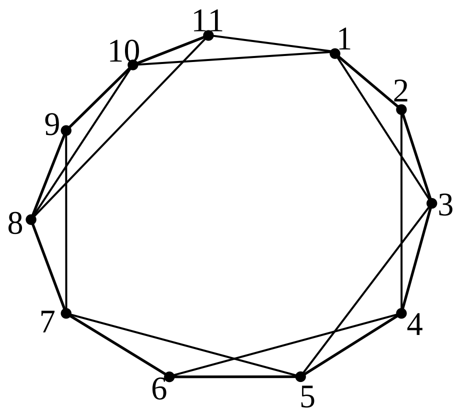

Figure 4: Circular – Arc Graph

Input: $I = \{1,2,3,4,5,6,7,8,9,10,11\}$

Step 1: NTRD={}

Step 2: $x = 1$

Step 3: $S = nrd^{+}[1] = \{1,2,3\}$

Step 4: $S_1 = nrd^{-}[1] = \{1,10,11\}$

Step 5: $S_2 = \{1,2,3\}$

Step 5.1: $a = max(S_1) = max(\{1,2,3\}) = 3$

Step 5.2: $\text{NTRD} = \text{NTRD} \cup \{3\} = \{3\}$

Step 6: $nrd(3)$ has no pendent vertex

$\text{NTRD} = \text{NTRD} \cup \{4\} = \{3,4\}$

Step 7: $d = max(\{3,4\}) = 4$

Step 8: $x = NI(4) = 7 \neq \emptyset \not\subseteq nrd^{-}(1)$

Step 2: $x = 7$

Step 3: $S = \{7,8,9\}$

Step 5: $S_2 = \{7,8,9\}$

Step 5.1: $a = max(\{7,8,9\} = 9$

Step 5.2: $\text{NTRD} = \{3,4\} \cup \{9\} = \{3,4,9\}$

Step 6: $nrd(9)$ has no pendent vertex

$\text{NTRD} = \{3,4,9\} \cup \{10\} = \{3,4,9,10\}$

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Step 7: $d = max(\{3,4,9,10\}) = 10$

Step 8: $x = NI(10) = \varnothing$

Step 9: End

Output: NTRD = $\{3,4,9,10\}$

Theorem 2.2: *Let G_i be a connected interval graph with the order $p \geq 6$, maximum degree $\Delta \leq p-2$, and minimum degree $\delta \geq 2$, then*

$$\gamma_{ntr} \leq p - \frac{\Delta}{2} - 1 = \phi(p,\Delta)$$

Proof: We prove by induction method on $l = p+q$

where, p is number of intervals and q is size of the graph.

We need to show that $\gamma_{ntr} \leq \phi(p,\Delta)$

Let $p \geq 6$ and $q \geq 7 \Rightarrow l \geq 13$

If $l = 13$, the interval graph G_i has two 3-cycles and so,

$$\gamma_{ntr}(G_i) = 2 = \phi(7,3) = \phi(p,\Delta)$$

Now, let $l \geq 14$ and $p' \geq 6$, $q' \geq 8$ and $\Delta' \geq 3$ be integer with $p'+q' \leq l$ and $\Delta' \leq p-2$.

For hypothesis of the induction, suppose all connected interval graphs G_i' of p' intervals and q' edges with the maximum degree Δ' and minimum degree $\delta \geq 2$ assure $\gamma_{ntr}(G_i') \leq \phi(p',\Delta')$.

Let $G_i = (V,E)$ be a connected interval graph of p intervals and q edges with $l = p+q$, maximum degree $\Delta \leq p-2$ and minimum degree $\delta \geq 2$.

Claim 1: *A connected proper interval sub-graph G_i' of G_i of p' intervals has maximum degree $\Delta \leq p'-2$ and minimum degree at least 2, and the sub-graph $G_i - V(G_i')$ contains no isolated vertices, then $\gamma_{ntr} \leq \phi(p,\Delta)$.*

Proof: The inductive hypothesis is satisfied by the graph G_i' have size q'. Then, $p'+q' \leq l$, and so G_i'.

Let $p' = p-k$ where $k \geq 0$.

By inductive hypothesis, $\gamma_{ntr}(G_i') \leq \phi(p',\Delta) = \phi(p-k,\Delta) = \phi(p,\Delta) - k$.

Hence, $\gamma_{ntr}(G_i) \leq \gamma_{ntr}(G_i') + k \leq \phi(p,\Delta)$, as desired.

Let, u be a vertex of maximum degree Δ in G_i, and let L be the set of all large vertices of G_i.

Claim 2: *Every two vertices in $L \setminus \{u\}$ have no common small neighbour.*

Claim 3: *Every vertex in* $L \setminus \{u\}$ *has a neighbour which is not adjacent to the vertex* u .

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