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Extension of Contraction Theorems in Intuitionistic Generalized Fuzzy Cone Metric Spaces

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ABSTRACT: This work aims to extend the idea of Banach contraction principle to intuitionistic generalized fuzzy cone metric spaces. We inherit the existing ideas to bring out new definitions in these spaces. This work extends the contractive conditions of self-mappings by having the presence of all the possible restrictions. This extension is up to the limit in a linear way it can be extended. We construct some contraction theorems for the generalized intuitionistic fuzzy cone metric spaces. We prove the existence of fixed points and their uniqueness of self-mappings under the extended contractive conditions.

Key words: Fixed point, Cone, Triangular, Fuzzy contractive, Symmetric.
Mathematics Subject Classification: 54H25, 47H10.

1. Introduction

In 1965, Zadeh [14], made a great contribution to the field of mathematics by proposing the concept of fuzzy sets. In the year 1986, Atanassov [13] extended these fuzzy sets to intuitionistic fuzzy sets which have made a definite change and promoted the field of applied research. These two ideas actually paved a great path that leads to several generalized metric spaces. Huang and Zhang [12] defined cone metric spaces which generalized the metric spaces by replacing the real numbers by an ordering Banach space. Tarkan Oner et. al. [11] introduced these spaces over fuzzy sets.

Mohamed and Ranjith [10] came up with intuitionistic fuzzy cone metric spaces in the year 2017. In 2019, Jeyaraman and Sowndrarajan [8] defined intuitionistic generalized fuzzy cone metric spaces and proved common fixed point theorems for (Φ, ψ)-weak contractions in these spaces. The idea of fuzzy contractive mapping was introduced by Gregori and Sapena [6], and, they have also extended the Banach's fixed point theorem with fuzzy contractive mappings. These ideas and results lead the way to extend the Banach contraction theorem in the intuitionistic generalized fuzzy cone metric spaces. We accomplish the work with new definitions which inherit the existing ideas.

2. Preliminaries

Definition 2.1 [12]

Let E be a real Banach space and $\mathcal{C}$ be a subset of E. $\mathcal{C}$ is called a cone if and only if

(i) $\mathcal{C}$ is closed, nonempty, and$i\mathcal{C} \neq 0$,
(ii) $a, b \in \mathbb{R}, a, b \geq 0, c_1, c_2 \in \mathcal{C} \implies ac_1 + bc_2 \in \mathcal{C}$,
(iii) $c \in \mathcal{C}$ and $-c \in \mathcal{C} \implies c = 0$.

The cones considered here are with nonempty interiors.

Definition 2.2 [8]

A 5-tuple $(X, \mathcal{M}, \mathcal{N} *, \diamond)$ is said to be an Intuitionistic Generalized Fuzzy Cone Metric Space, (briefly, IGFCMS), if $\mathcal{C}$ is a cone of E, X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is continuous t-conorm and $M, \mathcal{N}$ are fuzzy sets in $X^3 \times int(\mathcal{C})$ satisfying the following conditions:

For all $x, y, z, a \in X$ and $t, s \in int(\mathcal{C})$,

(i) $\mathcal{M}(\mathrm{x, y, z, t}) + \mathcal{N}(\mathrm{x, y, z, t}) \leq 1$,

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(ii) $\mathcal{M}(x,y,z,t) > 0$,
(iii) $\mathcal{M}(x,y,z,t) = 1 \Leftrightarrow x = y = z$,
(iv) $\mathcal{M}(x,y,z,t) = \mathcal{M}(p\{x,y,z\},t)$, where p is a permutation function,
(v) $\mathcal{M}(x,y,z,t+s) \geq \mathcal{M}(x,y,a,t) * \mathcal{M}(a,z,z,t)$,
(vi) $\mathcal{M}(x,y,z,.) : int(\mathcal{C}) \to (0,1]$ is continuous,
(vii) $\mathcal{N}(x,y,z,t) > 0$,
(viii) $\mathcal{N}(x,y,z,t) = 0 \Leftrightarrow x = y = z$,
(ix) $\mathcal{N}(x,y,z,t) = \mathcal{N}(p\{x,y,z\},t)$, where p is a permutation function
(x) $\mathcal{N}(x,y,z,t+s) \leq \mathcal{N}(x,y,a,t) \diamond \mathcal{N}(a,z,z,t)$,
(xi) $\mathcal{N}(x,y,z,.) : int(\mathcal{C}) \to (0,1]$ is continuous.

Then $(\mathcal{M},\mathcal{N})$ is called an intuitionistic generalized fuzzy cone metric on X. The functions $\mathcal{M}(x,y,z,t)$ and $\mathcal{N}(x,y,z,t)$ denote the degree of nearness and the degree of nonnearness between x, y and z with respect to t, respectively.

Example 2.3

Let $E = \mathbb{R}^2$ and consider the cone $\mathcal{C} = \{(c_1, c_2) \in \mathbb{R}^2 : c_1 \geq 0, c_2 \geq 0\}$ in E. Let $X = \mathbb{R}$ and the norms $*$ and $\diamond$ be defined by $a * b = ab$ and $a \diamond b = max\{a, b\}$. Define the functions $\mathcal{M}: X^3 \times int(\mathcal{C}) \longrightarrow [0,1]$ and $\mathcal{N}: X^3 \times int(\mathcal{C}) \longrightarrow [0,1]$ by

$$\mathcal{M}(x,y,z,t) = \frac{1}{e^{\frac{|x-y|+|y-z|+|z-x|}{\|t\|}}} \text{ and } \mathcal{N}(x,y,z,t) = \frac{e^{\frac{|x-y|+|y-z|+|z-x|}{\|t\|}}-1}{e^{\frac{|x-y|+|y-z|+|z-x|}{\|t\|}}},$$

for all $x, y, z \in X$ and $t \in int(\mathcal{C})$. Then $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ is an IGFCMS.

3. Main Results

Definition 3.1

An IGFCMS $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ is called a symmetric IGFCMS if, for all $x, y \in X$ and $t \in int(\mathcal{C})$, $\mathcal{M}$ and $\mathcal{N}$ satisfy the following conditions:
$\left(\frac{1}{\mathcal{M}(x,y,y,t)} - 1\right) = \left(\frac{1}{\mathcal{M}(y,x,x,t)} - 1\right)$ and $\mathcal{N}(x,y,y,t) = \mathcal{N}(y,x,x,t)$.

Definition 3.2

Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be an IGFCMS and $f: X \to X$ be a self-mapping. Then f is said to be a intuitionistic generalized fuzzy cone contractive if there exists $c \in (0,1)$ such that $\left(\frac{1}{\mathcal{M}(f(x),f(y),f(z),t)} - 1\right) \leq c\left(\frac{1}{\mathcal{M}(x,y,z,t)} - 1\right)$ and $\mathcal{N}(f(x), f(y), f(z), t) \leq c\mathcal{N}(x,y,z,t)$ for each $x, y, z \in X$ and $t \in int(\mathcal{C})$.

Definition 3.3

Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be an IGFCMS. $\mathcal{M}$ and $\mathcal{N}$ are said to be triangular if, for all $x, y, z, u \in X$ and $t \in int(\mathcal{C})$,
$\left(\frac{1}{\mathcal{M}(x,y,z,t)} - 1\right) \leq \left(\frac{1}{\mathcal{M}(x,y,u,t)} - 1\right) + \left(\frac{1}{\mathcal{M}(u,z,z,t)} - 1\right)$,
$\mathcal{N}(x,y,z,t) \leq \mathcal{N}(x,y,u,t) + \mathcal{N}(u,z,z,t)$.

Definition 3.4

Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be an IGFCMS. For $x \in X$, $r > 0$ and $t \in int(\mathcal{C})$, the open ball $\mathcal{B}_{\mathcal{C}}(x,r,t)$ with center at x and radius r is defined by
$\mathcal{B}_{\mathcal{C}}(x,r,t) = \{y \in X : \mathcal{M}(x,y,y,t) > 1 - r, \mathcal{N}(x,y,y,t) < r\}$.

Lemma 3.5 [4]

For each $c_1 \in int(\mathcal{C})$ and $c_2 \in int(\mathcal{C})$, there exists $c \in int(\mathcal{C})$ such that $c_1 - c \in int(\mathcal{C})$ and $c_2 - c \in int(\mathcal{C})$.

Theorem 3.6

Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be an IGFCMS. Then $\tau_{\mathcal{C}}$ defined below is a topology:
$\tau_{\mathcal{C}} = \{\mathcal{D} \subseteq X : x \in \mathcal{D} \Leftrightarrow \exists\, r \in (0,1), t \in int(\mathcal{C})\ such\ that\ \mathcal{B}_{\mathcal{C}}(x,r,t) \subset \mathcal{D}\}$.

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Proof

(i) It is obvious that $\emptyset \in \tau_{\mathcal{C}}$ and $X \in \tau_{\mathcal{C}}$.

(ii) Suppose $\mathcal{D}_1 \in \tau_{\mathcal{C}}$ and $\mathcal{D}_2 \in \tau_{\mathcal{C}}$ and $x \in \mathcal{D}_1 \cap \mathcal{D}_2$. Then $x \in \mathcal{D}_1$ and $x \in \mathcal{D}_2$.
$\Longrightarrow \exists\, r_{1,} r_2 \in (0,1)$ and $t_{1,} t_2 \in int(\mathrm{C})$ such that
$\mathcal{B}_{\mathcal{C}}(x, r_{1,}, t_{1,}) \subset \mathcal{D}_1$ and $\mathcal{B}_{\mathcal{C}}(x, r_2, t_2) \subset \mathcal{D}_2$.
By lemma 3.5, $\exists\, t \in int(\mathcal{C})$ such that $t_1 - t \in int(\mathcal{C})$, $t_2 - t \in int(\mathcal{C})$.
Let $r = min\{r_{1,} r_2\}$. Then $\mathcal{B}_{\mathcal{C}}(x, r, t) \subset \mathcal{B}_{\mathcal{C}}(x, r_{1,}, t_{1,}) \cap \mathcal{B}_{\mathcal{C}}(x, r_2, t_2) \subset \mathcal{D}_1 \cap \mathcal{D}_2$.
Hence $\mathcal{D}_1 \cap \mathcal{D}_2 \in \tau_{\mathcal{C}}$.

(iii) Let $\mathcal{D}_j \in \tau_{\mathcal{C}}$ for each $j \in J$, an index set and let $x \in \bigcup_{j \in J} \mathcal{D}_j$.
Then $x \in \mathcal{D}_{j_0}$ for some $j_0 \in J$.
$\Longrightarrow \exists\, r \in (0,1)$ and $t \in int(\mathcal{C})$ such that $\mathcal{B}_{\mathcal{C}}(x, r, t) \subset \mathcal{D}_{j_0}$.
As $\mathcal{D}_{j_0} \subset \bigcup_{j \in J} \mathcal{D}_j$, we have that $\mathcal{B}_{\mathcal{C}}(x, r, t) \subset \bigcup_{j \in J} \mathcal{D}_j$.
Thus $\bigcup_{j \in J} \mathcal{D}_j \in \tau_{\mathcal{C}}$.
From (i), (ii) and (iii), $\tau_{\mathcal{C}}$ is a topology.

Remark [11]

For any $r_1 > r_2$, there exists r_3 such that $r_1 * r_3 \geq r_2$ and for any r_4 there exists $r_5 \in (0,1)$ such that $r_5 * r_5 \geq r_4$, where $r_1, r_2, r_3, r_4, r_5 \in (0,1)$.

Theorem 3.7

Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be an IGFCMS. Then $(X, \tau_{\mathcal{C}})$ is Hausdorff.

Proof

Let $x, y \in X$ and $x \neq y$. Then $0 < \mathcal{M}(x, y, y, t) < 1$ and $0 < \mathcal{N}(x, y, y, t) < 1$.

Let $\mathcal{M}(x, y, y, t) = r_1$ and $\mathcal{N}(x, y, y, t) = r_2$. Take $r = max\{r_1, r_2\}$.

Now, for each $r_0 \in (r, 1)$, there exists $r_3, r_4 \in (0,1)$ such that

$$r_3 * r_3 \geq r_0 \text{ and } (1 - r_4) \diamond (1 - r_4) \leq 1 - r_0.$$

Let $r_5 = max\{r_3, r_4\}$. Suppose $\mathcal{B}_{\mathcal{C}}\left(x, 1 - r_{1,}, \frac{t}{2}\right) \cap \mathcal{B}_{\mathcal{C}}\left(y, 1 - r_2, \frac{t}{2}\right) \neq \emptyset$.

Then $\exists\, z \in \mathcal{B}_{\mathcal{C}}\left(x, 1 - r_{1,}, \frac{t}{2}\right) \cap \mathcal{B}_{\mathcal{C}}\left(y, 1 - r_2, \frac{t}{2}\right)$ and we have that

$$r_1 = \mathcal{M}(x, y, y, t) \geq \mathcal{M}\left(x, y, z, \frac{t}{2}\right) * \mathcal{M}\left(z, y, y, \frac{t}{2}\right)$$

$$\geq r_5 * r_5 \geq r_3 * r_3 \geq r_0 > r_1, \text{ and}$$

$$r_2 = \mathcal{N}(x, y, y, t) \leq \mathcal{N}\left(x, y, z, \frac{t}{2}\right) \diamond \mathcal{N}\left(z, y, y, \frac{t}{2}\right)$$

$$\leq (1 - r_5) \diamond (1 - r_5) \geq (1 - r_4) \diamond (1 - r_4) \geq (1 - r_0) < r_2.$$

This is a contradiction. Hence $\mathcal{B}_{\mathcal{C}}\left(x, 1 - r_{1,}, \frac{t}{2}\right) \cap \mathcal{B}_{\mathcal{C}}\left(y, 1 - r_2, \frac{t}{2}\right) = \emptyset$.
Therefore $(X, \tau_{\mathcal{C}})$ is Hausdorff.

Definition 3.8

Let $(X, \mathcal{M}, \mathcal{N} *, \diamond)$ be an IGFCMS, $x \in X$ and $\{x_n\}$ be a sequence in X.

(i) $\{x_n\}$ is said to converge to x if for all $t \in int(\mathcal{C})$, $\lim\limits_{n \to \infty} \left(\frac{1}{\mathcal{M}(x_n, x, x, t)} - 1\right) = 0$ and $\lim\limits_{n \to \infty} \mathcal{N}(x_n, x, x, t) = 0$. It is denoted by $\lim\limits_{n \to \infty} x_n = x$ or by $x_n \to x$ as $n \to \infty$.

(ii) $\{x_n\}$ is said to be a Cauchy sequence if for all $t \in int(\mathcal{C})$ and $m \in \mathbb{N}$, we have that

$$\lim_{n \to \infty} \left(\frac{1}{\mathcal{M}(x_{n+m}, x_n, x_n, t)} - 1\right) = 0 \text{ and } \lim_{n \to \infty} \mathcal{N}(x_{n+m}, x_n, x_n, t) = 0.$$

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(iii) $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ is called complete IGFCMS if every Cauchy sequence in X converges.

Remark

The convergence of sequences in an IGFCMS is considered in the sense of the topology defined here. Therefore, each converging sequence in an IGFCMS has a unique limit and this makes the definition of convergence meaningful.

Definition 3.9

Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be an IGFCMS. A sequence $\{x_n\}$ in X is fuzzy cone contractive if there exists $c \in (0,1)$ such that

$$\left(\frac{1}{\mathcal{M}(s_n, s_{n+1}, s_{n+1}, t)} - 1\right) \leq c\left(\frac{1}{\mathcal{M}(s_{n-1}, s_n, s_n, t)} - 1\right),$$

$$\mathcal{N}(s_n, s_{n+1}, s_{n+1}, t) \leq c\mathcal{N}(s_{n-1}, s_n, s_n, t)$$

for all $t \in int(\mathcal{C})$.

Lemma 3.10

An IGFCMS $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ is symmetric.

Proof

Let $\mathrm{x, y} \in \mathrm{X}$ and $\mathrm{t} \in \mathrm{int}(\mathcal{C})$. Then,

$$\lim_{r\to 0} \mathcal{M}(x, x, y, t + r) \geq \lim_{r\to 0}\big(\mathcal{M}(x, x, x, r) * \mathcal{M}(x, y, y, t)\big),$$
$$\lim_{r\to 0} \mathcal{M}(y, y, x, t + r) \geq \lim_{r\to 0}\big(\mathcal{M}(y, y, y, r) * \mathcal{M}(y, x, x, t)\big)$$
$$\Rightarrow \quad \mathcal{M}(x, x, y, t) \geq \mathcal{M}(x, y, y, t) \text{ and } \mathcal{M}(y, y, x, t) \geq \mathcal{M}(y, x, x, t)$$

$$\lim_{r\to 0} \mathcal{N}(x, x, y, t + r) \leq \lim_{r\to 0}\big(\mathcal{N}(x, x, x, r) \diamond \mathcal{N}(x, y, y, t)\big),$$
$$\lim_{r\to 0} \mathcal{N}(y, y, x, t + r) \leq \lim_{r\to 0}\big(\mathcal{N}(y, y, y, r) \diamond \mathcal{N}(y, x, x, t)\big)$$
$$\Rightarrow \quad \mathcal{N}(x, x, y, t) \leq \mathcal{N}(x, y, y, t) \text{ and } \mathcal{N}(y, y, x, t) \leq \mathcal{N}(y, x, x, t)$$

Hence $\mathcal{M}(x, y, y, t) = \mathcal{M}(y, x, x, t)$ and $\mathcal{N}(x, y, y, t) = \mathcal{N}(y, x, x, t)$.

Lemma 3.11

Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be an IGFCMS where $\mathcal{M}$ and $\mathcal{N}$ are triangular. Then any fuzzy cone contractive sequence in X is a Cauchy sequence.

Proof

Let the sequence $\{s_n\}$ be fuzzy cone contractive X. Then there exists $c \in (0,1)$ such that

$$\left(\frac{1}{\mathcal{M}(s_n, s_{n+1}, s_{n+1}, t)} - 1\right) \leq c\left(\frac{1}{\mathcal{M}(s_{n-1}, s_n, s_n, t)} - 1\right) \tag{3.11.1}$$

$$\mathcal{N}(s_n, s_{n+1}, s_{n+1}, t) \leq c\mathcal{N}(s_{n-1}, s_n, s_n, t) \tag{3.11.2}$$

Now, $\mathcal{M}$ and $\mathcal{N}$ are triangular. By Lemma 3.10, for $m > n > n_0$, $n_0 \in N$,

$$\left(\frac{1}{\mathcal{M}(s_n, s_n, s_m, t)} - 1\right) \leq \left(\left(\frac{1}{\mathcal{M}(s_n, s_n, s_{n+1}, t)} - 1\right) + \left(\frac{1}{\mathcal{M}(s_{n+1}, s_{n+1}, s_m, t)} - 1\right)\right)$$

$$\leq \left(\begin{array}{c}\left(\frac{1}{\mathcal{M}(s_n, s_n, s_{n+1}, t)} - 1\right) + \left(\frac{1}{\mathcal{M}(s_{n+1}, s_{n+1}, s_{n+2}, t)} - 1\right) \\ + \left(\frac{1}{\mathcal{M}(s_{n+2}, s_{n+2}, s_m, t)} - 1\right)\end{array}\right),$$

$$\mathcal{N}(s_n, s_n, s_m, t) \leq \mathcal{N}(s_n, s_n, s_{n+1}, t) + \mathcal{N}(s_{n+1}, s_{n+1}, s_m, t)$$

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$$\leq \begin{pmatrix} \mathcal{N}(s_n, s_n, s_{n+1}, t) + \mathcal{N}(s_{n+1}, s_{n+1}, s_{n+2}, t) \\ +\mathcal{N}(s_{n+2}, s_{n+2}, s_m, t) \end{pmatrix}.$$

Continuing the process, and, using (3.11.1) and (3.11.2), we finally arrive at

$$\left(\frac{1}{\mathcal{M}(s_n,s_n,s_m,t)} - 1\right) \leq \begin{pmatrix} \left(\frac{1}{\mathcal{M}(s_n,s_n,s_{n+1},t)} - 1\right) + \left(\frac{1}{\mathcal{M}(s_{n+1},s_{n+1},s_{n+2},t)} - 1\right) \\ + \cdots + \left(\frac{1}{\mathcal{M}(s_{m-1},s_{m-1},s_m,t)} - 1\right) \end{pmatrix}$$

$$\leq c^n \left(\frac{1}{\mathcal{M}(s_0,s_0,s_1,t)} - 1\right) + \cdots + c^{m-1} \left(\frac{1}{\mathcal{M}(s_0,s_0,s_1,t)} - 1\right)$$

$$= (c^n + \cdots + c^{m-1}) \left(\frac{1}{\mathcal{M}(s_0,s_0,s_1,t)} - 1\right)$$

$$\leq \frac{c^n}{1-c} \left(\frac{1}{\mathcal{M}(s_0,s_0,s_1,t)} - 1\right), \qquad (3.11.3)$$

$$\mathcal{N}(s_n, s_n, s_m, t) \leq \mathcal{N}(s_n, s_n, s_{n+1}, t) + \mathcal{N}(s_{n+1}, s_{n+1}, s_m, t) + \cdots + \mathcal{N}(s_{m-1}, s_{m-1}, s_m, t)$$

$$\leq c^n \mathcal{N}(s_0, s_0, s_1, t) + \cdots + c^{m-1} \mathcal{N}(s_0, s_0, s_1, t)$$

$$= (c^n + \cdots + c^{m-1}) \mathcal{N}(s_0, s_0, s_1, t)$$

$$\leq \frac{c^n}{1-c} \mathcal{N}(s_0, s_0, s_1, t) \qquad (3.11.4)$$

From (3.11.3) and (3.11.4), we have that

$$\left(\frac{1}{\mathcal{M}(s_n,s_n,s_m,t)} - 1\right) \to 0 \text{ and } \mathcal{N}(s_n, s_n, s_m, t) \to 0 \text{ as } n \to \infty.$$

Therefore $\{s_n\}$ is a Cauchy sequence.

Theorem 3.12

Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be a complete IGFCMS where $\mathcal{M}$ and $\mathcal{N}$ are triangular. If $\Gamma: X \to X$ is such that for all $x, y, z \in X$ and $t \in int(\mathcal{C})$,

$$\left(\frac{1}{\mathcal{M}(\Gamma x,\Gamma y,\Gamma z,t)} - 1\right) \leq \begin{Bmatrix} c_1 \left(\frac{1}{\mathcal{M}(x,y,z,t)} - 1\right) + c_2 \left(\frac{1}{\mathcal{M}(x,\Gamma x,\Gamma x,t)} - 1\right) + c_3 \left(\frac{1}{\mathcal{M}(y,\Gamma z,\Gamma z,t)} - 1\right) \\ +c_4 \left(\frac{1}{\mathcal{M}(y,\Gamma y,\Gamma y,t)} - 1\right) + c_5 \left(\frac{1}{\mathcal{M}(z,\Gamma z,\Gamma z,t)} - 1\right) \\ +c_6 \left(\frac{1}{\mathcal{M}(z,\Gamma y,\Gamma y,t)} - 1\right) + c_7 \left(\frac{1}{\mathcal{M}(y,\Gamma x,z,t)} - 1\right) \end{Bmatrix}$$

(3.12.1)

$$\mathcal{N}(\Gamma x, \Gamma y, \Gamma z, t) \leq \begin{Bmatrix} c_1 \mathcal{N}(x, y, z, t) + c_2 \mathcal{N}(x, \Gamma x, \Gamma x, t) + c_3 \mathcal{N}(y, \Gamma z, \Gamma z, t) \\ +c_4 \mathcal{N}(y, \Gamma y, \Gamma y, t) + c_5 \mathcal{N}(z, \Gamma z, \Gamma z, t) \\ +c_6 \mathcal{N}(z, \Gamma y, \Gamma y, t) + c_7 \mathcal{N}(y, \Gamma x, z, t) \end{Bmatrix}$$

(3.12.2)

where $c_i \in [0, +\infty]$, $i = 1, \dots, 6$ and $\sum_{i=1}^{6} c_i < 1$. Then Γ has a fixed point and such a point is unique if $c_1 + c_7 < 1$.

Proof

Let $s_0 \in X$ be arbitrary. Generate a sequence $\{s_n\}$ with $s_n = \Gamma s_{n-1}$ for $n \in \mathbb{N}$. If there exists a nonnegative integer m such that $s_{m+1} = s_m$, then $\Gamma s_m = s_m$ and s_m becomes a fixed point of Γ.

Suppose $s_n \neq s_{n-1}$ for any $n \in \mathbb{N}$.

From (3.12.1),

$$\left(\frac{1}{\mathcal{M}(s_n,s_{n+1},s_{n+1},t)} - 1\right) \leq \left(\frac{1}{\mathcal{M}(\Gamma s_{n-1},\Gamma s_n,\Gamma s_n,t)} - 1\right)$$

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$$\leq \left\{ \begin{array}{c} c_1\left(\frac{1}{\mathcal{M}(s_{n-1},s_n,s_n,t)}-1\right)+c_2\left(\frac{1}{\mathcal{M}(s_{n-1},\Gamma s_{n-1},\Gamma s_{n-1},t)}-1\right)+c_3\left(\frac{1}{\mathcal{M}(s_n,\Gamma s_n,\Gamma s_n,t)}-1\right) \\ +c_4\left(\frac{1}{\mathcal{M}(s_n,\Gamma s_n,\Gamma s_n,t)}-1\right)+c_5\left(\frac{1}{\mathcal{M}(s_n,\Gamma s_n,\Gamma s_n,t)}-1\right) \\ +c_6\left(\frac{1}{\mathcal{M}(s_n,\Gamma s_n,\Gamma s_n,t)}-1\right)+c_7\left(\frac{1}{\mathcal{M}(s_n,\Gamma s_{n-1},s_n,t)}-1\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} c_1\left(\frac{1}{\mathcal{M}(s_{n-1},s_n,s_n,t)}-1\right)+c_2\left(\frac{1}{\mathcal{M}(s_{n-1},s_n,s_n,t)}-1\right)+c_3\left(\frac{1}{\mathcal{M}(s_n,s_{n+1},s_{n+1},t)}-1\right) \\ +c_4\left(\frac{1}{\mathcal{M}(s_n,s_{n+1},s_{n+1},t)}-1\right)+c_5\left(\frac{1}{\mathcal{M}(s_n,s_{n+1},s_{n+1},t)}-1\right) \\ +c_6\left(\frac{1}{\mathcal{M}(s_n,s_{n+1},s_{n+1},t)}-1\right)+c_7\left(\frac{1}{\mathcal{M}(s_n,s_n,s_n,t)}-1\right) \end{array} \right\}$$

$$= \left\{(c_1+c_2)\left(\frac{1}{\mathcal{M}(s_{n-1},s_n,s_n,t)}-1\right)+(c_3+c_4+c_5+c_6)\left(\frac{1}{\mathcal{M}(s_n,s_{n+1},s_{n+1},t)}-1\right)\right\}.$$

Hence we have that $\left(\frac{1}{\mathcal{M}(s_n,s_{n+1},s_{n+1},t)}-1\right) \leq \frac{c_1+c_2}{1-(c_3+c_4+c_5+c_6)}\left(\frac{1}{\mathcal{M}(s_{n-1},s_n,s_n,t)}-1\right).$ (3.12.3)

Put $c=\frac{c_1+c_2}{1-(c_3+c_4+c_5+c_6)}$. Then $c<1$ and (3.12.3) becomes

$$\left(\frac{1}{\mathcal{M}(s_n,s_{n+1},s_{n+1},t)}-1\right) \leq c\left(\frac{1}{\mathcal{M}(s_{n-1},s_n,s_n,t)}-1\right). \qquad (3.12.4)$$

From (3.12.2),

$\mathcal{N}(s_n,s_{n+1},s_{n+1},t) \leq \mathcal{N}(\Gamma s_{n-1},\Gamma s_n,\Gamma s_n,t)$

$$\leq \left\{ \begin{array}{c} c_1\mathcal{N}(s_{n-1},s_n,s_n,t)+c_2\mathcal{N}(s_{n-1},s_n,s_n,t)+c_3\mathcal{N}(s_n,\Gamma s_n,\Gamma s_n,t) \\ +c_4\mathcal{N}(s_n,\Gamma s_n,\Gamma s_n,t)+c_5\mathcal{N}(s_n,\Gamma s_n,\Gamma s_n,t)+c_6\mathcal{N}(s_n,\Gamma s_n,\Gamma s_n,t) \\ +c_7\mathcal{N}(s_n,s_n,s_n,t) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} c_1\mathcal{N}(s_{n-1},s_n,s_n,t)+c_2\mathcal{N}(s_{n-1},s_n,s_n,t)+c_3\mathcal{N}(s_n,s_{n+1},s_{n+1},t) \\ +c_4\mathcal{N}(s_n,s_{n+1},s_{n+1},t)+c_5\mathcal{N}(s_n,s_{n+1},s_{n+1},t)+c_6\mathcal{N}(s_n,s_{n+1},s_{n+1},t) \\ +c_7\mathcal{N}(s_n,s_n,s_n,t) \end{array} \right\}$$

$$= \{(c_1+c_2)\mathcal{N}(s_{n-1},s_n,s_n,t)+(c_3+c_4+c_5+c_6)\mathcal{N}(s_n,s_{n+1},s_{n+1},t)\}.$$

Hence, we have that,

$$\mathcal{N}(s_n,s_{n+1},s_{n+1},t) \leq \frac{c_1+c_2}{1-(c_3+c_4+c_5+c_6)}\mathcal{N}(s_{n-1},s_n,s_n,t)$$

$$\Longrightarrow \mathcal{N}(s_n,s_{n+1},s_{n+1},t) \leq c\mathcal{N}(s_{n-1},s_n,s_n,t) \qquad (3.12.5)$$

(3.12.4) and (3.12.5) make the sequence $\{s_n\}$ is fuzzy cone contractive.

Hence by Lemma 3.11, $\{s_n\}$ is Cauchy in X.

As X is complete, there exists $\dot{s} \in X$ such that

$$\lim_{n\to\infty}\left(\frac{1}{\mathcal{M}(s_n,\dot{s},\dot{s},t)}-1\right)=0 \text{ and } \lim_{n\to\infty}\mathcal{N}(s_n,\dot{s},\dot{s},t)=0. \qquad (3.12.6)$$

By repeated application of (3.12.4) and (3.12.5), we obtain that

$$\left(\frac{1}{\mathcal{M}(s_n,s_{n+1},s_{n+1},t)}-1\right) \leq c^n\left(\frac{1}{\mathcal{M}(s_0,s_1,s_1,t)}-1\right),$$

$$\mathcal{N}(s_n,s_{n+1},s_{n+1},t) \leq c^n\mathcal{N}(s_0,s_1,s_1,t).$$

$$\Longrightarrow \lim_{n\to\infty}\left(\frac{1}{\mathcal{M}(s_n,s_{n+1},s_{n+1},t)}-1\right)=0 \text{ and } \lim_{n\to\infty}\mathcal{N}(s_n,s_{n+1},s_{n+1},t)=0. \qquad (3.12.7)$$

Now, $\left(\frac{1}{\mathcal{M}(s_{n+1},\Gamma\dot{s},\Gamma\dot{s},t)}-1\right)=\left(\frac{1}{\mathcal{M}(\Gamma s_n,\Gamma\dot{s},\Gamma\dot{s},t)}-1\right)$

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$$\leq \left\{ \begin{array}{c} c_1\left(\frac{1}{\mathcal{M}(s_n,\dot{s},\dot{s},t)}-1\right)+c_2\left(\frac{1}{\mathcal{M}(s_n,\Gamma s_n,\Gamma s_n,t)}-1\right)+c_3\left(\frac{1}{\mathcal{M}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t)}-1\right) \\ +c_4\left(\frac{1}{\mathcal{M}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t)}-1\right)+c_5\left(\frac{1}{\mathcal{M}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t)}-1\right) \\ +c_6\left(\frac{1}{\mathcal{M}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t)}-1\right)+c_7\left(\frac{1}{\mathcal{M}(\dot{s},\Gamma s_n,\dot{s},t)}-1\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} c_1\left(\frac{1}{\mathcal{M}(s_n,\dot{s},\dot{s},t)}-1\right)+c_2\left(\frac{1}{\mathcal{M}(s_n,s_{n+1},s_{n+1},t)}-1\right)+c_3\left(\frac{1}{\mathcal{M}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t)}-1\right) \\ +c_4\left(\frac{1}{\mathcal{M}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t)}-1\right)+c_5\left(\frac{1}{\mathcal{M}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t)}-1\right) \\ +c_6\left(\frac{1}{\mathcal{M}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t)}-1\right)+c_7\left(\frac{1}{\mathcal{M}(\dot{s},s_{n+1},\dot{s},t)}-1\right) \end{array} \right\}$$

$\longrightarrow d\left(\frac{1}{\mathcal{M}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t)}-1\right)$ as $n\to\infty$ where $d=c_3+c_4+c_5+c_6$,
since by (3.12.6) and (3.12.7).

Hence, $\lim\ sup_{n\to\infty}\left(\frac{1}{\mathcal{M}(s_{n+1},\Gamma\dot{s},\Gamma\dot{s},t)}-1\right)\leq\ d\left(\frac{1}{\mathcal{M}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t)}-1\right)$.

Similarly, $\lim\ sup_{n\to\infty}\mathcal{N}(s_{n+1},\Gamma\dot{s},\Gamma\dot{s},t)\leq\ d\mathcal{N}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t)$.

As $\mathcal{M}$ and $\mathcal{N}$ are triangular,

$$\left(\frac{1}{\mathcal{M}(\Gamma\dot{s},\Gamma\dot{s},\dot{s},t)}-1\right)\leq\left(\frac{1}{\mathcal{M}(\Gamma\dot{s},\Gamma\dot{s},s_{n+1},t)}-1\right)+\left(\frac{1}{\mathcal{M}(s_{n+1},\dot{s},,\dot{s},t)}-1\right), \tag{3.12.8}$$

$$\mathcal{N}(\Gamma\dot{s},\Gamma\dot{s},\dot{s},t)\leq\mathcal{N}(\Gamma\dot{s},\Gamma\dot{s},s_{n+1},t)+\mathcal{N}(s_{n+1},\dot{s},,\dot{s},t). \tag{3.12.9}$$

From (3.12.6) to (3.10.9), we can bring that

$$\left(\frac{1}{\mathcal{M}(\Gamma\dot{s},\Gamma\dot{s},\dot{s},t)}-1\right)\leq d\left(\frac{1}{\mathcal{M}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t)}-1\right), \mathcal{N}(\dot{s},\dot{s},\Gamma\dot{s},t)\leq d\mathcal{N}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t).$$

$$\Rightarrow\left(\frac{1}{\mathcal{M}(\Gamma\dot{s},\Gamma\dot{s},\dot{s},t)}-1\right)=0,\ \mathcal{N}(\Gamma\dot{s},\Gamma\dot{s},\dot{s},t)=0 \text{ since } d<1.$$

$$\Rightarrow \Gamma\dot{s}=\dot{s}.$$

Thus, we can conclude that $\dot{s}$ is a fixed point of Γ.

Suppose $\Gamma\ddot{s}=\ddot{s}$. Then from (3.12.1),

$$\left(\frac{1}{\mathcal{M}(\Gamma\dot{s},\Gamma\ddot{s},\Gamma\ddot{s},t)}-1\right)\leq \left\{ \begin{array}{c} c_1\left(\frac{1}{\mathcal{M}(\dot{s},\ddot{s},\ddot{s},t)}-1\right)+c_2\left(\frac{1}{\mathcal{M}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t)}-1\right)+c_3\left(\frac{1}{\mathcal{M}(\ddot{s},\Gamma\ddot{s},\Gamma\ddot{s},t)}-1\right) \\ +c_4\left(\frac{1}{\mathcal{M}(\ddot{s},\Gamma\ddot{s},\Gamma\ddot{s},t)}-1\right)+c_5\left(\frac{1}{\mathcal{M}(\ddot{s},\Gamma\ddot{s},\Gamma\ddot{s},t)}-1\right) \\ +c_6\left(\frac{1}{\mathcal{M}(\ddot{s},\Gamma\ddot{s},\Gamma\ddot{s},t)}-1\right)+c_7\left(\frac{1}{\mathcal{M}(\ddot{s},\Gamma\dot{s},\ddot{s},t)}-1\right) \end{array} \right\}.$$

$$\Rightarrow\left(\frac{1}{\mathcal{M}(\dot{s},\ddot{s},\ddot{s},t)}-1\right)\leq(c_1+c_7)\left(\frac{1}{\mathcal{M}(\dot{s},\ddot{s},\ddot{s},t)}-1\right).$$

$$\Rightarrow\left(\frac{1}{\mathcal{M}(\dot{s},\ddot{s},\ddot{s},t)}-1\right)=0 \text{ if } c_1+c_7<1.$$

Similarly, $\mathcal{N}(\dot{s},\ddot{s},\ddot{s},t)=0$.

Hence, we can conclude that Γ has a unique fixed point if $c_1+c_7<1$.

Example 3.13

Let $X=[0,\infty)$ with metric d defined by $d(x,y)=|x-y|$ for all $x,y\in X$ and let $\mathcal{C}=\mathbb{R}^+$.
Define the t-norm $*$ and the t-conorm $\diamond$ by $a*b=min\{a,b\}$ and $a\diamond b=max\{a,b\}$. Define the $\mathcal{M}$ and $\mathcal{N}$ by

$$\mathcal{M}(x,y,z,t)=\frac{\mathrm{t}}{\mathrm{t}+l(|x-y|+|y-z|+|z-x|)} \text{ and } \mathcal{N}(x,y,z,t)=\frac{l(|x-y|+|y-z|+|z-x|)}{\mathrm{t}+l(|x-y|+|y-z|+|z-x|)}$$

for all $x,y,z\in X$ and $t\in int(\mathcal{C})$ where $l=10$.

Then it is clear that $(X,\mathcal{M},\mathcal{N},*,\diamond)$ is a complete IGFCMS and that $\mathcal{M}$ and $\mathcal{N}$ are triangular.

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Consider the self-map $\Gamma: X \longrightarrow X$ given by $\Gamma x = \begin{cases} \frac{5}{4}x + 3, & x \in [0,1], \\ \frac{3}{4}x + \frac{7}{2}, & x \in [1, \infty). \end{cases}$

Then $\left(\frac{1}{\mathcal{M}(\Gamma x, \Gamma y, \Gamma z, t)} - 1\right) = \frac{5}{4}\left(\frac{1}{\mathcal{M}(x,y,z,t)} - 1\right)$ and $\mathcal{N}(\Gamma x, \Gamma y, \Gamma z, t) \geq \frac{5}{4}\mathcal{N}(x,y,z,t)$when $x, y, z \in [0,1]$. Hence Γ is not fuzzy cone contractive. Therefore, we cannot use the contraction theorem to assure the existence of fixed points. But here Γ satisfies the conditions (3.12.1) and (3.12.2) with $c_1 = \frac{3}{80}, c_2 = \frac{17}{80}, c_3 = c_4 = c_5 = \frac{1}{20}$,
$c_6 = 0, c_7 = \frac{1}{20}$. Therefore Γ has a unique fixed point and this point is $x = 14$.

Corollary 3.14

Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$be a complete IGFCMS where $\mathcal{M}$ and $\mathcal{N}$ are triangular. If $\Gamma: X \to X$ is such that for all $x, y, z \in X$, $t \in int(\mathcal{C})$,

$$\left(\frac{1}{\mathcal{M}(\Gamma x, \Gamma y, \Gamma z, t)} - 1\right) \leq \left\{\begin{matrix} c_1\left(\frac{1}{\mathcal{M}(x,y,z,t)} - 1\right) + c_2\left(\frac{1}{\mathcal{M}(x,\Gamma x,\Gamma x,t)} - 1\right) + \\ c_3\left(\frac{1}{\mathcal{M}(y,\Gamma z,\Gamma z,t)} - 1\right) + c_4\left(\frac{1}{\mathcal{M}(y,\Gamma x,z,t)} - 1\right) \end{matrix}\right\}$$

$$\mathcal{N}(\Gamma x, \Gamma y, \Gamma z, t) \leq \left\{\begin{matrix} c_1\mathcal{N}(x,y,z,t) + c_2\mathcal{N}(x,\Gamma x,\Gamma x,t) + \\ c_3\mathcal{N}(y,\Gamma z,\Gamma z,t) + c_4\mathcal{N}(y,\Gamma x,z,t) \end{matrix}\right\}$$

where $c_i \in [0, +\infty]$, $i = 1, \ldots, 4$ and $c_1 + c_2 + c_3 < 1$. Then Γ has a fixed point and such a point is unique if $c_1 + c_4 < 1$.

Corollary 3.15

Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$be a complete IGFCMS where $\mathcal{M}$ and $\mathcal{N}$ are triangular. If $\Gamma: X \to X$ is such that for all $x, y, z \in X$, $t \in int(\mathcal{C})$,

$$\left(\frac{1}{\mathcal{M}(\Gamma x, \Gamma y, \Gamma z, t)} - 1\right) \leq \left\{\begin{matrix} c_1\left(\frac{1}{\mathcal{M}(x,y,z,t)} - 1\right) + c_2\left(\frac{1}{\mathcal{M}(x,\Gamma x,\Gamma x,t)} - 1\right) + c_3\left(\frac{1}{\mathcal{M}(y,\Gamma z,\Gamma z,t)} - 1\right) + \\ c_4\left(\frac{1}{\mathcal{M}(y,\Gamma y,\Gamma y,t)} - 1\right) + c_5\left(\frac{1}{\mathcal{M}(z,\Gamma z,\Gamma z,t)} - 1\right) + c_6\left(\frac{1}{\mathcal{M}(z,\Gamma y,\Gamma y,t)} - 1\right) \end{matrix}\right\},$$

$$\mathcal{N}(\Gamma x, \Gamma y, \Gamma z, t) \leq \left\{\begin{matrix} c_1\mathcal{N}(x,y,z,t) + c_2\mathcal{N}(x,\Gamma x,\Gamma x,t) + c_3\mathcal{N}(y,\Gamma z,\Gamma z,t) + \\ c_4\mathcal{N}(y,\Gamma y,\Gamma y,t) + c_5\mathcal{N}(z,\Gamma z,\Gamma z,t) + c_6\mathcal{N}(z,\Gamma y,\Gamma y,t) \end{matrix}\right\}$$

where $c_i \in [0, +\infty]$, $i = 1, \ldots, 6$ and $\sum_{i=1}^{6} c_i < 1$. Then Γ has a unique fixed point.

Corollary 3.16

Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be a complete IGFCMS where $\mathcal{M}$ and $\mathcal{N}$ are triangular. If $\Gamma: X \to X$ satisfies (3.12.1) and (3.12.2) with $\sum_{i=1}^{7} c_i < 1$, then Γ has a unique fixed point.

The following theorem gives the generalized contractive condition which considers all the possible restrictions.

Theorem 3.17

Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be a complete IGFCMS where $\mathcal{M}$ and $\mathcal{N}$ are triangular. If $\Gamma: X \to X$ is such that for all $x, y, z \in X$, $t \in int(\mathcal{C})$,

$$\left(\frac{1}{\mathcal{M}(\Gamma x, \Gamma y, \Gamma z, t)} - 1\right) \leq \left\{\begin{matrix} c_1\left(\frac{1}{\mathcal{M}(x,y,z,t)} - 1\right) + c_2\left(\frac{1}{\mathcal{M}(x,\Gamma x,z,t)} - 1\right) + c_3\left(\frac{1}{\mathcal{M}(x,x,\Gamma x,t)} - 1\right) + \\ c_4\left(\frac{1}{\mathcal{M}(x,\Gamma x,\Gamma x,t)} - 1\right) + c_5\left(\frac{1}{\mathcal{M}(y,\Gamma y,z,t)} - 1\right) + c_6\left(\frac{1}{\mathcal{M}(y,\Gamma z,z,t)} - 1\right) + \\ c_7\left(\frac{1}{\mathcal{M}(\Gamma x,\Gamma y,z,t)} - 1\right) + c_8\left(\frac{1}{\mathcal{M}(\Gamma x,\Gamma z,y,t)} - 1\right) + c_9\left(\frac{1}{\mathcal{M}(y,y,\Gamma y,t)} - 1\right) + \\ c_{10}\left(\frac{1}{\mathcal{M}(z,z,\Gamma z,t)} - 1\right) + c_{11}\left(\frac{1}{\mathcal{M}(y,\Gamma y,\Gamma y,t)} - 1\right) + c_{12}\left(\frac{1}{\mathcal{M}(z,\Gamma z,\Gamma z,t)} - 1\right) + \\ c_{13}\left(\frac{1}{\mathcal{M}(z,\Gamma y,\Gamma y,t)} - 1\right) + c_{14}\left(\frac{1}{\mathcal{M}(y,\Gamma z,\Gamma z,t)} - 1\right) \\ c_{15}\left(\frac{1}{\mathcal{M}(\Gamma y,\Gamma z,x,t)} - 1\right) + c_{16}\left(\frac{1}{\mathcal{M}(x,\Gamma z,\Gamma z,t)} - 1\right) \end{matrix}\right\},$$

(3.17.1)

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$$\mathcal{N}(\Gamma x,\Gamma y,\Gamma z,t) \leq \left\{\begin{array}{c} c_1\mathcal{N}(x,y,z,t)+c_2\mathcal{N}(x,\Gamma x,z,t)+c_3\mathcal{N}(x,x,\Gamma x,t)+ \\ c_4\mathcal{N}(x,\Gamma x,\Gamma x,t)+c_5\mathcal{N}(y,\Gamma y,z,t)+c_6\mathcal{N}(y,\Gamma z,z,t)+ \\ c_7\mathcal{N}(\Gamma x,\Gamma y,z,t)+c_8\mathcal{N}(\Gamma x,\Gamma z,y,t)+c_9\mathcal{N}(y,y,\Gamma y,t) \\ +c_{10}\mathcal{N}(z,z,\Gamma z,t)+c_{11}\mathcal{N}(y,\Gamma y,\Gamma y,t)+c_{12}\mathcal{N}(z,\Gamma z,\Gamma z,t) \\ +c_{13}\mathcal{N}(z,\Gamma y,\Gamma y,t)+c_{14}\mathcal{N}(y,\Gamma z,\Gamma z,t) \\ +c_{15}\mathcal{N}(\Gamma y,\Gamma z,x,t)+c_{16}\mathcal{N}(x,\Gamma z,\Gamma z,t) \end{array}\right\},$$

(3.17.2)

where $c_i \in [0,+\infty], i = 1,\dots,16$ and $c_1+\cdots+c_{14}+2(c_{15}+c_{16})<1$. Then Γ has a unique fixed point.

Proof

Let $s_0 \in X$ be arbitrary. Generate a sequence $\{s_n\}$ with $s_n = \Gamma s_{n-1}$ for $n \in \mathbb{N}$. If there exists a nonnegative integer m such that $s_{m+1} = s_m$, then $\Gamma s_m = s_m$ and s_m becomes a fixed point of Γ.

Suppose $s_n \neq s_{n-1}$ for any $n \in \mathbb{N}$.

As $\mathcal{M}$ and $\mathcal{N}$ are triangular and by Lemma (3.10),

$$\left(\frac{1}{\mathcal{M}(s_{n+1},s_{n+1},s_{n-1},t)}-1\right)=\left(\frac{1}{\mathcal{M}(s_{n-1},s_{n-1},s_{n+1},t)}-1\right)$$
$$\leq \left(\frac{1}{\mathcal{M}(s_{n-1},s_{n-1},s_n,t)}-1\right)+\left(\frac{1}{\mathcal{M}(s_n,s_{n+1},s_{n+1},t)}-1\right). \quad (3.17.3)$$

$$\mathcal{N}(s_{n+1},s_{n+1},s_{n-1},t) \leq \mathcal{N}(s_{n-1},s_{n-1},s_{n+1},t)$$
$$\leq \mathcal{N}(s_{n-1},s_{n-1},s_n,t)+\mathcal{N}(s_n,s_{n+1},s_{n+1},t). \quad (3.17.4)$$

$$\left(\frac{1}{\mathcal{M}(s_{n-1},s_n,s_{n+1},t)}-1\right) \leq \left(\frac{1}{\mathcal{M}(s_{n-1},s_n,s_n,t)}-1\right)+\left(\frac{1}{\mathcal{M}(s_n,s_{n+1},s_{n+1},t)}-1\right). \quad (3.17.5)$$

$$\mathcal{N}(s_{n-1},s_n,s_{n+1},t) \leq \mathcal{N}(s_{n-1},s_n,s_n,t)+\mathcal{N}(s_n,s_{n+1},s_{n+1},t). \quad (3.17.6)$$

Using (3.17.1) and (3.17.2) as in Theorem (3.12), together with (3.17.3) to (3.17.6), we arrive at

$$\left(\frac{1}{\mathcal{M}(s_n,s_{n+1},s_{n+1},t)}-1\right) \leq \frac{c_1+\cdots+c_4+c_{15}+c_{16}}{1-(c_5+\cdots+c_{16})}\left(\frac{1}{\mathcal{M}(s_{n-1},s_n,s_n,t)}-1\right),$$

$$\mathcal{N}(s_n,s_{n+1},s_{n+1},t) \leq \frac{c_1+\cdots+c_4+c_{15}+c_{16}}{1-(c_5+\cdots+c_{16})}\mathcal{N}(s_{n-1},s_n,s_n,t).$$

Putting $c = \frac{c_1+\cdots+c_4+c_{15}+c_{16}}{1-(c_5+\cdots+c_{16})}$, the above inequality becomes

$$\left(\frac{1}{\mathcal{M}(s_n,s_{n+1},s_{n+1},t)}-1\right) \leq c\left(\frac{1}{\mathcal{M}(s_{n-1},s_n,s_n,t)}-1\right), \quad (3.17.7)$$

$$\mathcal{N}(s_n,s_{n+1},s_{n+1},t) \leq c\mathcal{N}(s_{n-1},s_n,s_n,t). \quad (3.17.8)$$

(3.17.5) and (3.17.6) made the sequence $\{s_n\}$ fuzzy cone contractive.

Hence by Lemma (3.11), $\{s_n\}$ is Cauchy in X.

As X is complete, there exists $\dot{s} \in X$ such that

$$\lim_{n\to\infty}\left(\frac{1}{\mathcal{M}(s_n,\dot{s},\dot{s},t)}-1\right)=0 \text{ and } \lim_{n\to\infty}\mathcal{N}(s_n,\dot{s},\dot{s},t)=0. \quad (3.17.9)$$

By repeated application of (3.17.7) and (3.17.8), we obtain that

$$\left(\frac{1}{\mathcal{M}(s_n,s_{n+1},s_{n+1},t)}-1\right) \leq c^n\left(\frac{1}{\mathcal{M}(s_0,s_1,s_1,t)}-1\right),$$

$$\mathcal{N}(s_n,s_{n+1},s_{n+1},t) \leq c^n\mathcal{N}(s_0,s_1,s_1,t),$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{1}{\mathcal{M}(s_n,s_{n+1},s_{n+1},t)}-1\right)=0 \text{ and } \lim_{n\to\infty}\mathcal{N}(s_n,s_{n+1},s_{n+1},t)=0. \quad (3.17.10)$$

From (3.17.1),

$$\left(\frac{1}{\mathcal{M}(s_{n+1},\Gamma\dot{s},\Gamma\dot{s},t)}-1\right)=\left(\frac{1}{\mathcal{M}(\Gamma s_n,\Gamma\dot{s},\Gamma\dot{s},t)}-1\right)$$
$$\leq d\left(\frac{1}{\mathcal{M}(\Gamma\dot{s},\Gamma\dot{s},\dot{s},t)}-1\right) \text{ where } d = c_5+\cdots+c_{16}.$$

Hence, $\lim\ sup_{n\to\infty}\left(\frac{1}{\mathcal{M}(s_{n+1},\Gamma\dot{s},\Gamma\dot{s},t)}-1\right)\leq\ d\left(\frac{1}{\mathcal{M}(\Gamma\dot{s},\Gamma\dot{s},\dot{s},t)}-1\right)$.

Similarly, $\lim\ sup_{n\to\infty}\mathcal{N}(s_{n+1},\Gamma\dot{s},\Gamma\dot{s},t)\leq d\mathcal{N}(\Gamma\dot{s},\Gamma\dot{s},\dot{s},t)$.

As $\mathcal{M}$ and $\mathcal{N}$ are triangular,

$$\left(\frac{1}{\mathcal{M}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t)}-1\right)\leq\left(\frac{1}{\mathcal{M}(\Gamma\dot{s},\Gamma\dot{s},s_{n+1},t)}-1\right)+\left(\frac{1}{\mathcal{M}(s_{n+1},\dot{s},\dot{s},t)}-1\right), \tag{3.17.11}$$

$$\mathcal{N}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t)\leq\mathcal{N}(\Gamma\dot{s},\Gamma\dot{s},s_{n+1},t)+\mathcal{N}(s_{n+1},\dot{s},\dot{s},t). \tag{3.17.12}$$

From (3.17.9), to (3.17.12), we can bring that

$$\left(\frac{1}{\mathcal{M}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t)}-1\right)\leq d\left(\frac{1}{\mathcal{M}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t)}-1\right), \mathcal{N}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t)\leq d\mathcal{N}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t).$$

$$\Rightarrow\left(\frac{1}{\mathcal{M}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t)}-1\right)=0,\ \mathcal{N}(\dot{s},\Gamma\dot{s},\Gamma\dot{s},t)=0 \text{ as } d<1.$$

$$\Rightarrow\Gamma\dot{s}=\dot{s}.$$

Thus, we can conclude that $\dot{s}$ is a fixed point of Γ.

Suppose $\Gamma\ddot{s}=\ddot{s}$. Then from (3.17.1), (3.17.2) and by Lemma (3.10),

$$\left(\frac{1}{\mathcal{M}(\dot{s},\ddot{s},\ddot{s},t)}-1\right)\leq d'\left(\frac{1}{\mathcal{M}(\dot{s},\ddot{s},\ddot{s},t)}-1\right),$$
$$\mathcal{N}(\dot{s},\ddot{s},\ddot{s},t)\leq d'\mathcal{N}(\dot{s},\ddot{s},\ddot{s},t),$$

where $d'=c_1+c_2+c_7+c_8+c_{15}+c_{16}$. These inequalities imply that

$$\left(\frac{1}{\mathcal{M}(\dot{s},\ddot{s},\ddot{s},t)}-1\right)=0, \mathcal{N}(\dot{s},\ddot{s},\ddot{s},t)=0 \text{ since } d'<1.$$

Thus, we can conclude that Γ has a unique fixed point.

Corollary 3.18

Let $(X,\mathcal{M},\mathcal{N},*,\diamond)$ be a complete IGFCMS where $\mathcal{M}$ and $\mathcal{N}$ are triangular. If $\Gamma: X\to X$ is such that for all $x,y,z\in X$, $t\in int(\mathcal{C})$,

$$\left(\frac{1}{\mathcal{M}(\Gamma x,\Gamma y,\Gamma z,t)}-1\right)\leq\left\{\begin{array}{c} c_1\left(\frac{1}{\mathcal{M}(x,y,z,t)}-1\right)+c_2\left(\frac{1}{\mathcal{M}(x,\Gamma x,z,t)}-1\right)+c_3\left(\frac{1}{\mathcal{M}(\Gamma x,\Gamma y,z,t)}-1\right)\\ +c_4\left(\frac{1}{\mathcal{M}(y,\Gamma z,\Gamma z,t)}-1\right)+c_5\left(\frac{1}{\mathcal{M}(\Gamma y,\Gamma z,x,t)}-1\right)+c_6\left(\frac{1}{\mathcal{M}(x,\Gamma y,z,t)}-1\right)\end{array}\right\},$$

$$\mathcal{N}(\Gamma x,\Gamma y,\Gamma z,t)\leq\left\{\begin{array}{l} c_1\mathcal{N}(x,y,z,t)+c_2\mathcal{N}(x,\Gamma x,z,t)+c_3\mathcal{N}(\Gamma x,\Gamma y,z,t)+\\ c_4\mathcal{N}(y,\Gamma z,\Gamma z,t)+c_5\mathcal{N}(\Gamma y,\Gamma z,x,t)+c_6\mathcal{N}(x,\Gamma y,z,t)\end{array}\right\}$$

where $c_i\in[0,+\infty]$, $i=1,\ldots,6$ and $c_1+c_2+c_3+c_4+2(c_5+c_6)<1$. Then Γ has a unique fixed point.

Conclusion

This work provided new definitions by inheriting some existing ideas to the intuitionistic generalized fuzzy cone metric space. We constructed some fixed point theorems as an extension of Banach contraction theorem by giving a general form of contractive conditions for self-mappings under which these mappings have fixed points.

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