JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1343-1353
https://publishoa.com
ISSN: 1309-3452

Some New Stability Results of Quadratic Functional Equation In Neutrosophic Normed Spaces

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ABSTRACT
In this paper, we determine some stability results concerning the 2-dimensional vector variable quadratic functional equation $f(x+y,z+w)+f(x-y,z-w)=2f(x,z)+2f(y,w)$ in Neutrosophic Normed Spaces. We define the Neutrosophic Continuity of the 2-dimensional vector variable quadratic mapping and prove that the existence of a solution for any approximately 2-dimensional vector variable quadratic mapping implies the completeness of Neutrosophic Normed Space.
Keywords:
Neutrosophic Normed Space, Stability, Banach Space, Continuity.
AMS Subject Classification (2020): 40A30 ; 40G15.

1. Introduction

Fuzzy set theory is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. It has also very useful applications in various fields. The fuzzy topology proves to be a very useful tool to deal with such situations where the use of classical theories breaks down. In 1984, Katsaras [5] defined fuzzy norms on linear spaces and at the same year Wu and Fang also introduced a notion of fuzzy normed spaces and gave the generalization of the Kolmogoroff normalized theorem for fuzzy topological linear spaces. There are many situations where the norm of a vector is not possible to find and the concept of Neutrosophic Norm [9,11,12] seems to be more suitable in such cases, that is, we can deal with such situations by modeling the inexactness through the intuitionistic fuzzy norm. Stability problem of a functional equation was first posed by Ulam [15] which was generalized by Aoki [1] and Rassias [10] for additive mappings and linear mappings, respectively.

The concept Neutrosophy implies impartial knowledge of thought and then neutral describes the basic difference between neutral, fuzzy, intuitive fuzzy set and logic. The Neutrosophic Set (NS) was investigated by Smarandache [13] who defined the degree of indeterminacy (I) as independent component. In [14], Neutrosophic logic was firstly examined. It is a logic where each proposition is determined to have a degree of truth (T), falsity (F) and indeterminacy (I). A Neutrosophic Set (NS) is determined as a set where every component of the universe has a degree of T, F and I.

In IFSs the 'degree of non-belongingness' is not independent but it is dependent on the 'degree of belongingness'. FSs can be thought as a remarkable case of an IFS where the 'degree of non-belongingness' of an element is absolutely equal to '1-degree of belongingness'. Uncertainty is based on the belongingness degree in IFSs, whereas the uncertainty in NS is considered independently from T and F values. Since no any limitations among the degree of T, F, I, NSs are actually more general than IFS. Some notable features of NMS have been examined.

Neutrosophic Normed Space (NNS) and statistical convergence in NNS has been investigated by Kirisci and Simsek [8]. Neutrosophic Set (NS) and Neutrosophic logic has used by applied sciences and theoretical science such as decision making, robotics, summability theory.

The stability problem for the 2-dimensional vector variable quadratic functional equation was proved by the authors [2] for mappings $f: X\times X\to Y,$ where X is a real normed space and Y is a Banach space. In this paper, we determine some stability results concerning the 2-dimensional vector variable quadratic functional equation
$f(x+y,z+w)+f(x-y,z-w)=2f(x,z)+2f(y,w)$ in NNS. We define the Neutrosophic Continuity of the 2-dimensional vector variable quadratic mappings and prove that the existence of a solution for any approximately 2-dimensional vector variable quadratic mapping implies the completeness of NNS

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2. Preliminaries

Definition 2.1:

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A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is said to be continuous t-norm if it satisfies the following conditions;

i. $*$ is associative and commutative,
ii. $*$ is continuous,
iii. $a * 1 = a$, for all $a \in [0,1]$,
iv. $a * b \le c * d$ whenever $a \le c$ and $b \le d$, for each $a, b, c, d \in [0,1]$.

Definition 2.2:

A binary operation $\diamond: [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-conorm if it satisfies the following conditions;

i. $\diamond$ is associative and commutative,
ii. $\diamond$ is continuous,
iii. $a \diamond 0 = a$, for all $a \in [0,1]$,
iv. $a \diamond b \le c \diamond d$ whenever $a \le c$ and $b \le d$ for each $a, b, c, d \in [0,1]$.

Definition 2.3:

The seven-tuple $(X, \mu, \vartheta, \omega, *, \diamond, \odot)$ is said to be Neutrosophic Normed Space (NNS) if X is a vector space, $*$ is a continuous t-norm, $\diamond$ and $\odot$ are continuous t-conorm and μ, ϑ, ω are fuzzy sets on $X \times \mathbb{R}$ satisfying the following conditions; For every $x, y \in X$ and $s, t > 0$

i. $\mu(x,t) + \vartheta(x,t) + \omega(x,t) \le 3$,
ii. $0 \le \mu(x,t) \le 1, 0 \le \vartheta(x,t) \le 1, 0 \le \omega(x,t) \le 1$,
iii. $\mu(x,t) > 0$,
iv. $\mu(x,t) = 1$ iff $x = 0$,
v. $\mu(\alpha x, t) = \mu\left(x, \frac{t}{|\alpha|}\right)$, for each $\alpha \ne 0$,
vi. $\mu(x,t) * \mu(y,s) \le \mu(x+y, t+s)$,
vii. $\mu(x,\cdot): (0,\infty) \to [0,1]$ is continuous,
viii. $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$,
ix. $\vartheta(x,t) < 1$
x. $\vartheta(x,t) = 0$ iff $x = 0$,
xi. $\vartheta(\alpha x, t) = \vartheta\left(x, \frac{t}{|\alpha|}\right)$, for each $\alpha \ne 0$,
xii. $\vartheta(x,t) \diamond \vartheta(y,s) \ge \vartheta(x+y, t+s)$,
xiii. $\vartheta(x,\cdot): (0,\infty) \to [0,1]$ is continuous,
xiv. $\lim_{t\to\infty} \vartheta(x,t) = 0$ and $\lim_{t\to 0} \vartheta(x,t) = 1$.
xv. $\omega(x,t) < 1$,
xvi. $\omega(x,t) = 0$ iff $x = 0$,
xvii. $\omega(\alpha x, t) = \omega\left(x, \frac{t}{|\alpha|}\right)$,for each $\alpha \ne 0$,
xviii. $\omega(x,t) \odot \omega(y,s) \ge \omega(x+y, t+s)$,
xix. $\omega(x,\cdot): (0,\infty) \to [0,1]$ is continuous,
xx. $\lim_{t\to\infty} \omega(x,t) = 0$ and $\lim_{t\to 0} \omega(x,t) = 1$.

Example 2.4:

Let $(X, \|\cdot\|)$ be a normed space, $a * b = ab$, $a \diamond b = min\{a+b, 1\}$ and $a \odot b = min\{a+b, 1\}$, for all $a, b \in [0,1]$. For all $x \in X$ and every $t > 0$, Consider $\mu(x,t) = \begin{Bmatrix} \frac{t}{t+\|x\|} & \text{if } t > 0 \\ 0 & \text{if } t \le 0 \end{Bmatrix}$, $\vartheta(x,t) = \begin{Bmatrix} \frac{\|x\|}{t+\|x\|} & \text{if } t > 0 \\ 1 & \text{if } t \le 0 \end{Bmatrix}$ and $\omega(x,t) = \begin{Bmatrix} \frac{\|x\|}{t} & \text{if } t > 0 \\ 1 & \text{if } t \le 0 \end{Bmatrix}$. Then $(X, \mu, \vartheta, \omega, *, \diamond, \odot)$ is NNS.

Remark 2.5:

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In NNS $(X,\mu,\vartheta,\omega,*,\diamond,\odot)$, $\mu(x,\cdot)$ is non-decreasing $\vartheta(x,\cdot)$ and $\omega(x,\cdot)$ are non-increasing, for all $x\in X$.

Definition 2.6:

Let $(X,\mu,\vartheta,\omega,*,\diamond,\odot)$ be a NNS. A sequence $x=(x_k)$ is said to be Neutrosophic Convergent to $L\in X$ if $\lim_{k\to\infty}\mu(x_k-L,t)=1$, $\lim_{k\to\infty}\vartheta(x_k-L)=0$ and $\lim_{k\to\infty}\omega(x_k-L)=0$, for all $t>0$.
In this case we write $x_k\to L$ as $k\to\infty$.

Definition 2.7:

A sequence $x=(x_k)$ is said to be Neutrosophic Cauchy Sequence if $\lim_{k\to\infty}\mu\big(x_{k+p}-x_k,t\big)=1$, $\lim_{k\to\infty}\vartheta\big(x_{k+p}-x_k,t\big)=0$ and $\lim_{k\to\infty}\omega\big(x_{k+p}-x_k,t\big)=0$, for all $p\in\mathbb{N}$ and $t>0$.

Definition 2.8:

The NNS $(X,\mu,\vartheta,\omega,*,\diamond,\odot)$ is said to be complete if every Neutrosophic Cauchy sequence in $(X,\mu,\vartheta,\omega,*,\diamond,\odot)$ is Neutrosophic Convergent in $(X,\mu,\vartheta,\omega,*,\diamond,\odot)$ is also called Neutrosophic Banach Space.

3. Neutrosophic Stability

Consider the functional equation

$$f(x+y,z+w)+f(x-y,z-w)=2f(x,z)+2f(y,w) \tag{3.1}$$

having quadratic forms $f(x,y)=ax^2+bxy+cy^2$ as solutions.
We begin with a generalized Hyers-Ulam type theorem in NNS for (3.1).

Theorem 3.1:

Let X be a linear space and let $(Z,\mu',\vartheta',\omega')$ be NNS. Let $\varphi: X\times X\times X\times X\to Z$ be a mapping such that for some $0<\alpha<4$,

$$\left\{\begin{array}{l}\mu'(\varphi(2x,2y,2z,2w),t)\geq\mu'(\alpha\varphi(x,y,z,w),t),\\ \vartheta'(\varphi(2x,2y,2z,2w),t)\leq\vartheta'(\alpha\varphi(x,y,z,w),t)\text{ and}\\ \omega'(\varphi(2x,2y,2z,2w),t)\leq\omega'(\alpha\varphi(x,y,z,w),t).\end{array}\right\} \tag{3.1.1}$$

$\lim_{n\to\infty}\mu'(\varphi(2^nx,2^ny,2^nz,2^nw),4^nt)=1$, $\lim_{n\to\infty}\vartheta'(\varphi(2^nx,2^ny,2^nz,2^nw),4^nt)=0$ and $\lim_{n\to\infty}\omega'(\varphi(2^nx,2^ny,2^nz,2^nw),4^nt)=0$, for all $x,y,z,w\in X$ and all $t>0$.
Let (Y,μ,ϑ,ω) be NBS and let $f: X\times X\to Y$ be a mapping such that

$$\left\{\begin{array}{l}\mu(f(x+y,z+w)+f(x-y,z-w)-2f(x,z)-2f(y,w),t)\geq\mu'(\varphi(x,y,z,w),t),\\ \vartheta(f(x+y,z+w)+f(x-y,z-w)-2f(x,z)-2f(y,w),t)\leq\vartheta'(\varphi(x,y,z,w),t)\text{ and}\\ \omega(f(x+y,z+w)+f(x-y,z-w)-2f(x,z)-2f(y,w),t)\leq\omega'(\varphi(x,y,z,w),t),\end{array}\right\} \tag{3.1.2}$$

for all $x,y,z,w\in X$ and all $t>0$. Then there exists a unique mapping $F: X\times X\to Y$ satisfying (3.1) such that

$$\left\{\begin{array}{l}\mu\left(F(x,y)-f(x,y)+\frac{1}{3}f(0,0),t\right)\geq*^{\infty}\mu'(\varphi(x,x,y,y),(4-\alpha)t),\\ \vartheta\left(F(x,y)-f(x,y)+\frac{1}{3}f(0,0),t\right)\leq\ {}^{\infty}\vartheta'(\varphi(x,x,y,y),(4-\alpha)t)\text{ and}\\ \omega\left(F(x,y)-f(x,y)+\frac{1}{3}f(0,0),t\right)\leq\odot^{\infty}\omega'(\varphi(x,x,y,y),(4-\alpha)t),\end{array}\right\} \tag{3.1.3}$$

for all $x,y\in X$ and all $t>0$, where $*^{\infty}=a*a*\ldots$, $\diamond^{\infty}=a\diamond a\diamond\ldots$ and $\odot^{\infty}=a\odot a\odot\ldots$, for all $a\in[0,1]$.

Proof:

Putting $y=x$ and $w=z$ in (3.1.2) we get,

$$\left\{\begin{array}{l}\mu\left(\frac{f(2x,2y)+f(0,0)}{4}-f(x,z),\frac{t}{4}\right)\geq\mu'(\varphi(x,x,z,z),t),\\ \vartheta\left(\frac{f(2x,2y)+f(0,0)}{4}-f(x,z),\frac{t}{4}\right)\leq\vartheta'(\varphi(x,x,z,z),t)\text{ and}\\ \omega\left(\frac{f(2x,2y)+f(0,0)}{4}-f(x,z),\frac{t}{4}\right)\leq\omega'(\varphi(x,x,z,z),t),\end{array}\right\} \tag{3.1.4}$$

for all $x,z\in X$ and all $t>0$. Replacing x by 2^nx and z by 2^nz in (3.1.4) and using (3.1.1) we get,

$\mu\left(\frac{f(2^{n+1}x,2^{n+1}z)+f(0,0)}{4^{n+1}}-\frac{f(2^nx,2^nz)}{4^n},\frac{t}{4^{n+1}}\right)\geq\mu'(\varphi(2^nx,2^nx,2^nz,2^nz),t)\geq\mu'\left(\varphi(x,x,z,z),\frac{t}{\alpha^n}\right)$,
$\vartheta\left(\frac{f(2^{n+1}x,2^{n+1}z)+f(0,0)}{4^{n+1}}-\frac{f(2^nx,2^nz)}{4^n},\frac{t}{4^{n+1}}\right)\leq\vartheta'(\varphi(2^nx,2^nx,2^nz,2^nz),t)\leq\vartheta'\left(\varphi(x,x,z,z),\frac{t}{\alpha^n}\right)$ and
$\omega\left(\frac{f(2^{n+1}x,2^{n+1}z)+f(0,0)}{4^{n+1}}-\frac{f(2^nx,2^nz)}{4^n},\frac{t}{4^{n+1}}\right)\leq\omega'(\varphi(2^nx,2^nx,2^nz,2^nz),t)\leq\omega'\left(\varphi(x,x,z,z),\frac{t}{\alpha^n}\right)$, for all $x,z\in X$, all $n\geq 0$ and all $t>0$. By replacing t by α^nt we have

$$\left\{\begin{array}{l} \mu\left(\frac{f(2^{n+1}x,2^{n+1}z)+f(0,0)}{4^{n+1}}-\frac{f(2^nx,2^nz)}{4^n},\frac{\alpha^n t}{4^{n+1}}\right)\geq \mu'(\varphi(x,x,z,z),t), \\ \vartheta\left(\frac{f(2^{n+1}x,2^{n+1}z)+f(0,0)}{4^{n+1}}-\frac{f(2^nx,2^nz)}{4^n},\frac{\alpha^n t}{4^{n+1}}\right)\leq \vartheta'(\varphi(x,x,z,z),t) \text{ and} \\ \omega\left(\frac{f(2^{n+1}x,2^{n+1}z)+f(0,0)}{4^{n+1}}-\frac{f(2^nx,2^nz)}{4^n},\frac{\alpha^n t}{4^{n+1}}\right)\leq \omega'(\varphi(x,x,z,z),t), \end{array}\right\} \quad (3.1.5)$$

for all $x,z\in X$, all $n\geq 0$ and all $t>0$. It follows from,
$\sum_{k=0}^{n-1}\left[\frac{f(2^{k+1}x,2^{k+1}z)+f(0,0)}{4^{k+1}}-\frac{f(2^kx,2^kz)}{4^k}\right]=\frac{f(2^nx,2^nz)}{4^n}-f(x,z)+\frac{1}{3}\left(1-\frac{1}{4^n}\right)f(0,0)$ and (3.1.5) that,

$$\left\{\begin{array}{c} \mu\left(\frac{f(2^nx,2^nz)}{4^n}-f(x,z)+\frac{1}{3}\left(1-\frac{1}{4^n}\right)f(0,0),\sum_{k=0}^{n-1}\frac{\alpha^k t}{4^{k+1}}\right) \\ \geq \prod_{k=0}^{n-1}\mu\left(\frac{f(2^{k+1}x,2^{k+1}z)+f(0,0)}{4^{k+1}}-\frac{f(2^kx,2^kz)}{4^k},\frac{\alpha^k t}{4^{k+1}}\right) \\ \geq *^n\,\mu'(\varphi(x,x,z,z),t), \\ \vartheta\left(\frac{f(2^nx,2^nz)}{4^n}-f(x,z)+\frac{1}{3}\left(1-\frac{1}{4^n}\right)f(0,0),\sum_{k=0}^{n-1}\frac{\alpha^k t}{4^{k+1}}\right) \\ \leq \coprod_{k=0}^{n-1}\vartheta\left(\frac{f(2^{k+1}x,2^{k+1}z)+f(0,0)}{4^{k+1}}-\frac{f(2^kx,2^kz)}{4^k},\frac{\alpha^k t}{4^{k+1}}\right) \\ \leq \diamondsuit^n\,\vartheta'(\varphi(x,x,z,z),t) \text{ and} \\ \omega\left(\frac{f(2^nx,2^nz)}{4^n}-f(x,z)+\frac{1}{3}\left(1-\frac{1}{4^n}\right)f(0,0),\sum_{k=0}^{n-1}\frac{\alpha^k t}{4^{k+1}}\right) \\ \leq \coprod_{k=0}^{n-1}\omega\left(\frac{f(2^{k+1}x,2^{k+1}z)+f(0,0)}{4^{k+1}}-\frac{f(2^kx,2^kz)}{4^k},\frac{\alpha^k t}{4^{k+1}}\right) \\ \leq \odot^n\,\omega'(\varphi(x,x,z,z),t), \end{array}\right\} \quad (3.1.6)$$

for all $x,z\in X$, for all $n\in\mathbb{N}$ and all $t>0$, where $\prod_{j=1}^n a_j=a_1*a_2*\ldots*a_n$, $\coprod_{j=1}^n a_j=a_1\diamondsuit a_2\diamondsuit\ldots\diamondsuit a_n$, $\coprod_{j=1}^n a_j=a_1\odot a_2\odot\ldots\odot a_n$, $*^n a=\prod_{j=1}^n a=a*a*\ldots*a$, $\diamondsuit^n a=\coprod_{j=1}^n a=a\diamondsuit a\diamondsuit\ldots\diamondsuit a$ and $\odot^n a=\coprod_{j=1}^n a=a\odot a\odot\ldots\odot a$, for all $a,a_1,a_2,\ldots,a_n\in[0,1]$.
By replacing x with 2^mx and z with 2^mz in (3.1.6) we have,

$$\mu\left(\frac{f(2^{n+m}x,2^{n+m}z)}{4^{n+m}}-\frac{f(2^mx,2^mz)}{4^m}+\frac{1}{3.4^m}\left(1-\frac{1}{4^n}\right)f(0,0),\sum_{k=0}^{n-1}\frac{\alpha^k t}{4^{k+m+1}}\right)$$
$$\geq *^n\,\mu'(\varphi(2^mx,2^mx,2^mz,2^mz),t)\geq *^n\,\mu'\left(\varphi(x,x,z,z),\frac{t}{\alpha^m}\right),$$
$$\vartheta\left(\frac{f(2^{n+m}x,2^{n+m}z)}{4^{n+m}}-\frac{f(2^mx,2^mz)}{4^m}+\frac{1}{3.4^m}\left(1-\frac{1}{4^n}\right)f(0,0),\sum_{k=0}^{n-1}\frac{\alpha^k t}{4^{k+m+1}}\right)$$
$$\leq \diamondsuit^n\,\vartheta'(\varphi(2^mx,2^mx,2^mz,2^mz),t)\leq \diamondsuit^n\,\vartheta'\left(\varphi(x,x,z,z),\frac{t}{\alpha^m}\right) \text{ and}$$
$$\omega\left(\frac{f(2^{n+m}x,2^{n+m}z)}{4^{n+m}}-\frac{f(2^mx,2^mz)}{4^m}+\frac{1}{3.4^m}\left(1-\frac{1}{4^n}\right)f(0,0),\sum_{k=0}^{n-1}\frac{\alpha^k t}{4^{k+m+1}}\right)$$
$$\leq \odot^n\,\omega'(\varphi(2^mx,2^mx,2^mz,2^mz),t)\leq \odot^n\,\omega'\left(\varphi(x,x,z,z),\frac{t}{\alpha^m}\right),$$

for all $x,z\in X$, all $m\geq 0$, $n\in\mathbb{N}$ and all $t>0$. Replacing t by $\frac{t}{\sum_{k=m}^{n+m-1}\frac{\alpha^k}{4^{k+1}}}$ we get,

$$\left\{\begin{array}{l} \mu\left(\frac{f(2^{n+m}x,2^{n+m}z)}{4^{n+m}}-\frac{f(2^m x,2^m z)}{4^m}+\frac{1}{3.4^m}\left(1-\frac{1}{4^n}\right)f(0,0),t\right) \\ \geq *^n\ \mu'\left(\varphi(x,x,z,z),\frac{t}{\sum_{k=m}^{n+m-1}\frac{\alpha^k}{4^{k+1}}}\right), \\ \vartheta\left(\frac{f(2^{n+m}x,2^{n+m}z)}{4^{n+m}}-\frac{f(2^m x,2^m z)}{4^m}+\frac{1}{3.4^m}\left(1-\frac{1}{4^n}\right)f(0,0),t\right) \\ \leq \diamond^n\ \vartheta'\left(\varphi(x,x,z,z),\frac{t}{\sum_{k=m}^{n+m-1}\frac{\alpha^k}{4^{k+1}}}\right) \text{ and} \\ \omega\left(\frac{f(2^{n+m}x,2^{n+m}z)}{4^{n+m}}-\frac{f(2^m x,2^m z)}{4^m}+\frac{1}{3.4^m}\left(1-\frac{1}{4^n}\right)f(0,0),t\right) \\ \leq \odot^n\ \omega'\left(\varphi(x,x,z,z),\frac{t}{\sum_{k=m}^{n+m-1}\frac{\alpha^k}{4^{k+1}}}\right), \end{array}\right\} \quad (3.1.7)$$

for all $x,z \in X$, all $m \geq 0$, all $n \in \mathbb{N}$ and all $t > 0$.

Since $0 < \alpha < 4$, $\sum_{k=0}^{\infty}\left(\frac{\alpha}{4}\right)^k < \infty$ and $\sum_{k=m}^{n+m-1}\left(\frac{\alpha^k}{4^{k+1}}\right)^k \to 0$ as $m \to \infty$, for all $n \in \mathbb{N}$.

Thus $\frac{t}{\sum_{k=m}^{n+m-1}\frac{\alpha^k}{4^{k+1}}} \to \infty$ and $\mu'\left(\varphi(x,x,z,z),\frac{t}{\sum_{k=m}^{n+m-1}\frac{\alpha^k}{4^{k+1}}}\right) \to 1$ as $m \to \infty$, for all $x,z \in X$ all $n \in \mathbb{N}$ and all $t > 0$.

Hence the Cauchy criterion for convergence in NNS. Shows that $\frac{f(2^n x,2^n z)}{4^n}$ is a Cauchy sequence in (Y,μ,ϑ,ω), for all $x,z \in X$. Since (Y,μ,ϑ,ω) is complete, this sequence converges to some point $F(x,z) \in Y$, for all $x,z \in X$.

Put $m = 0$ in (3.1.7) to obtain,

$\mu\left(\frac{f(2^n x,2^n z)}{4^n} - f(x,z) + \frac{1}{3}\left(1-\frac{1}{4^n}\right)f(0,0),t\right) \geq *^n\ \mu'\left(\varphi(x,x,z,z),\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{4^{k+1}}}\right)$,

$\vartheta\left(\frac{f(2^n x,2^n z)}{4^n} - f(x,z) + \frac{1}{3}\left(1-\frac{1}{4^n}\right)f(0,0),t\right) \leq \diamond^n\ \vartheta'\left(\varphi(x,x,z,z),\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{4^{k+1}}}\right)$ and

$\omega\left(\frac{f(2^n x,2^n z)}{4^n} - f(x,z) + \frac{1}{3}\left(1-\frac{1}{4^n}\right)f(0,0),t\right) \leq \odot^n\ \omega'\left(\varphi(x,x,z,z),\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{4^{k+1}}}\right)$,

for all $x,z \in X$, all $n \in \mathbb{N}$ and all $t > 0$. Taking the limit as $n \to \infty$ and using the definition of NNS we get,

$$\mu\left(F(x,y) - f(x,y) + \frac{1}{3}f(0,0),t\right) \geq *^\infty\ \mu'(\varphi(x,x,y,y),(4-\alpha)t),$$
$$\vartheta\left(F(x,y) - f(x,y) + \frac{1}{3}f(0,0),t\right) \leq \diamond^\infty\ \vartheta'(\varphi(x,x,y,y),(4-\alpha)t) \text{ and}$$
$$\omega\left(F(x,y) - f(x,y) + \frac{1}{3}f(0,0),t\right) \leq \odot^\infty\ \omega'(\varphi(x,x,y,y),(4-\alpha)t),$$

for all $x,y \in X$, and all $t > 0$.

Replacing x,y,z,w and t in (3.1.2) by $2^n x, 2^n y, 2^n z, 2^n w$ and $4^n t$ respectively we have,

$\mu\left(\frac{f(2^n x+2^n y,2^n z+2^n w)}{4^n} + \frac{f(2^n x-2^n y,2^n z-2^n w)}{4^n} - 2\frac{f(2^n x,2^n z)}{4^n} - 2\frac{f(2^n y,2^n w)}{4^n},t\right)$
$\geq \mu'(\varphi(2^n x,2^n x,2^n y,2^n y),4^n t)$,

$\vartheta\left(\frac{f(2^n x+2^n y,2^n z+2^n w)}{4^n} + \frac{f(2^n x-2^n y,2^n z-2^n w)}{4^n} - 2\frac{f(2^n x,2^n z)}{4^n} - 2\frac{f(2^n y,2^n w)}{4^n},t\right)$
$\leq \vartheta'(\varphi(2^n x,2^n x,2^n y,2^n y),4^n t)$ and

$\omega\left(\frac{f(2^n x+2^n y,2^n z+2^n w)}{4^n} + \frac{f(2^n x-2^n y,2^n z-2^n w)}{4^n} - 2\frac{f(2^n x,2^n z)}{4^n} - 2\frac{f(2^n y,2^n w)}{4^n},t\right)$
$\leq \omega'(\varphi(2^n x,2^n x,2^n y,2^n y),4^n t)$ for all $x,y,z,w \in X$, all $n \in \mathbb{N}$ and all $t > 0$.

$\lim_{n\to\infty}\mu'(\varphi(2^n x,2^n x,2^n y,2^n y),4^n t) = 1$, $\lim_{n\to\infty}\vartheta'(\varphi(2^n x,2^n x,2^n y,2^n y),4^n t) = 0$ and $\lim_{n\to\infty}\omega'(\varphi(2^n x,2^n x,2^n y,2^n y),4^n t) = 0$, for all $x,y, \in X$ and all $t > 0$. We observe that F fulfills(3.1).

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https://publishoa.com
ISSN: 1309-3452

To prove the uniqueness of the mapping F, assume that there exists a mapping $G: X \times X \to Y$ which also satisfies (3.1) and (3.1.3). For fix $x, y \in X$, clearly $F(2^n x, 2^n y) = 4^n F(x, y)$ and $G(2^n x, 2^n y) = 4^n G(x, y)$, for all $n \in \mathbb{N}$. It follows from (3.1.3) that,

$\mu(F(x,y) - G(x,y), t) = \mu\left(\frac{F(2^n x, 2^n y)}{4^n} - \frac{G(2^n x, 2^n z)}{4^n}, t\right)$
$\geq \mu\left(\frac{F(2^n x, 2^n y)}{4^n} - \frac{f(2^n x, 2^n y)}{4^n} + \frac{1}{3.4^n} f(0,0), \frac{t}{2}\right) * \mu\left(-\frac{G(2^n x, 2^n y)}{4^n} + \frac{f(2^n x, 2^n y)}{4^n} - \frac{1}{3.4^n} f(0,0), \frac{t}{2}\right) \geq$
$*^2 *^\infty \mu'\left(\varphi(2^n x, 2^n x, 2^n y, 2^n y), \frac{4^n(4-\alpha)t}{2}\right) \geq *^2 *^\infty \mu'\left(\varphi(x, x, y, y), \frac{4^n(4-\alpha)t}{2\alpha^n}\right)$,

$\vartheta(F(x,y) - G(x,y), t) = \vartheta\left(\frac{F(2^n x, 2^n y)}{4^n} - \frac{G(2^n x, 2^n z)}{4^n}, t\right)$
$\leq \vartheta\left(\frac{F(2^n x, 2^n y)}{4^n} - \frac{f(2^n x, 2^n y)}{4^n} + \frac{1}{3.4^n} f(0,0), \frac{t}{2}\right) \diamond \vartheta\left(-\frac{G(2^n x, 2^n y)}{4^n} + \frac{f(2^n x, 2^n y)}{4^n} - \frac{1}{3.4^n} f(0,0), \frac{t}{2}\right) \leq$
$\diamond^2 \diamond^\infty \vartheta'\left(\varphi(2^n x, 2^n x, 2^n y, 2^n y), \frac{4^n(4-\alpha)t}{2}\right) \leq \diamond^2 \diamond^\infty \vartheta'\left(\varphi(x, x, y, y), \frac{4^n(4-\alpha)t}{2\alpha^n}\right)$ and

$\omega(F(x,y) - G(x,y), t) = \omega\left(\frac{F(2^n x, 2^n y)}{4^n} - \frac{G(2^n x, 2^n z)}{4^n}, t\right)$
$\leq \omega\left(\frac{F(2^n x, 2^n y)}{4^n} - \frac{f(2^n x, 2^n y)}{4^n} + \frac{1}{3.4^n} f(0,0), \frac{t}{2}\right) \odot \omega\left(-\frac{G(2^n x, 2^n y)}{4^n} + \frac{f(2^n x, 2^n y)}{4^n} - \frac{1}{3.4^n} f(0,0), \frac{t}{2}\right) \leq$
$\odot^2 \odot^\infty \vartheta'\left(\varphi(2^n x, 2^n x, 2^n y, 2^n y), \frac{4^n(4-\alpha)t}{2}\right) \leq \odot^2 \odot^\infty \vartheta'\left(\varphi(x, x, y, y), \frac{4^n(4-\alpha)t}{2\alpha^n}\right)$, for all $n \in \mathbb{N}$ and all $t > 0$.

Since $\lim_{n\to\infty} \frac{4^n(4-\alpha)t}{2\alpha^n} = \infty$, for all $t > 0$ we get,
$\lim_{n\to\infty} \mu'\left(\varphi(x, x, y, y), \frac{4^n(4-\alpha)t}{2\alpha^n}\right) = 1$, $\lim_{n\to\infty} \vartheta'\left(\varphi(x, x, y, y), \frac{4^n(4-\alpha)t}{2\alpha^n}\right) = 0$
and $\lim_{n\to\infty} \omega'\left(\varphi(x, x, y, y), \frac{4^n(4-\alpha)t}{2\alpha^n}\right) = 0$, for all $t > 0$.
Therefore $\mu(F(x,y) - G(x,y), t) = 1$, $\vartheta(F(x,y) - G(x,y), t) = 0$ and
$\omega(F(x,y) - G(x,y), t) = 0$, for all $t > 0$. Hence $F(x,y) = G(x,y)$.

Example 3.2:

Let X be a Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and Z be a normed space. Denote by (μ, ϑ, ω) and $(\mu', \vartheta', \omega')$ be the Neutrosophic Norms given as in Example (2.4) on X and Z respectively. Let $\|\cdot\|$ be the induced norm on X by the inner product $\langle \cdot,\cdot \rangle$ on X.
Let $\varphi: X \times X \times X \times X \to Z$ be a mapping defined by
$\varphi: (x, y, z, w) = 2(\|x\| + \|y\| + \|z\| + \|w\|) z_0$, for all $x, y, z, w \in X$, where z_0 is a fixed unit vector in Z.
Define a mapping $f: X \times X \to X$ by $f(x,y) = \langle x, y + x_0 \rangle x_0$, $x, y \in X$, where x_0 is a fixed unit vector in X. Then
$\mu(f(x+y, z+w) + f(x-y, z-w) - 2f(x,z) - 2f(y,w), t) = \mu(-2\langle y, x_0 \rangle x_0, t)$
$= \frac{t}{t + 2|\langle y, x_0 \rangle|} \geq \frac{t}{t + 2\|y\|} \geq \frac{t}{t + 2(\|x\| + \|y\| + \|z\| + \|w\|)} = \mu'(\varphi(x, y, z, w), t)$,
$\vartheta(f(x+y, z+w) + f(x-y, z-w) - 2f(x,z) - 2f(y,w), t) = \vartheta(-2\langle y, x_0 \rangle x_0, t)$
$= \frac{2|\langle y, x_0 \rangle|}{t + 2|\langle y, x_0 \rangle|} \leq \frac{2\|y\|}{t + 2\|y\|} \leq \frac{2(\|x\| + \|y\| + \|z\| + \|w\|)}{t + 2(\|x\| + \|y\| + \|z\| + \|w\|)} = \vartheta'(\varphi(x, y, z, w), t)$ and
$\omega(f(x+y, z+w) + f(x-y, z-w) - 2f(x,z) - 2f(y,w), t) = \mu(-2\langle y, x_0 \rangle x_0, t)$
$= \frac{2|\langle y, x_0 \rangle|}{t} \leq \frac{2\|y\|}{t} \leq \frac{2(\|x\| + \|y\| + \|z\| + \|w\|)}{t} = \omega'(\varphi(x, y, z, w), t)$, for all $x, y, z, w \in X$ and all $t > 0$. Also,
$\mu'(\varphi(2x, 2y, 2z, 2w), t) = \frac{t}{t + 4(\|x\| + \|y\| + \|z\| + \|w\|)} = \mu'(2\varphi(x, y, z, w), t)$,
$\vartheta'(\varphi(2x, 2y, 2z, 2w), t) = \frac{4(\|x\| + \|y\| + \|z\| + \|w\|)}{t + 4(\|x\| + \|y\| + \|z\| + \|w\|)} = \vartheta'(2\varphi(x, y, z, w), t)$ and
$\omega'(\varphi(2x, 2y, 2z, 2w), t) = \frac{4(\|x\| + \|y\| + \|z\| + \|w\|)}{t} = \omega'(2\varphi(x, y, z, w), t)$,
for all $x, y, z, w \in X$ and all $t > 0$. Thus,
$\lim_{n\to\infty} \mu'(\varphi(2^n x, 2^n y, 2^n z, 2^n w), 4^n t) = \lim_{n\to\infty} \frac{4^n t}{4^n t + 2^{n+1}(\|x\| + \|y\| + \|z\| + \|w\|)} = 1$,
$\lim_{n\to\infty} \vartheta'(\varphi(2^n x, 2^n y, 2^n z, 2^n w), 4^n t) = \lim_{n\to\infty} \frac{2^{n+1}(\|x\| + \|y\| + \|z\| + \|w\|)}{4^n t + 2^{n+1}(\|x\| + \|y\| + \|z\| + \|w\|)} = 0$ and
$\lim_{n\to\infty} \omega'(\varphi(2^n x, 2^n y, 2^n z, 2^n w), 4^n t) = \lim_{n\to\infty} \frac{2^{n+1}(\|x\| + \|y\| + \|z\| + \|w\|)}{4^n t} = 0$, for all $x, y, z, w \in X$ and all $t > 0$.
Hence the conditions of Theorem (3.1) for $\alpha = 2$ are fulfilled.
Therefore, there is a unique quadratic mapping $F: X \times X \to X$ such that
$\mu(F(x,y) - f(x,y), t) \geq \mu'\big((4(\|x\| + \|y\|) z_0, 2t\big)$,
$\vartheta(F(x,y) - f(x,y), t) \leq \vartheta'\big((4(\|x\| + \|y\|) z_0, 2t\big)$ and
$\omega(f(x,y) - f(x,y), t) \leq \omega'\big((4(\|x\| + \|y\|) z_0, 2t\big)$, for all $x, y, z, w \in X$, and all $t > 0$.

Theorem 3.3:

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https://publishoa.com
ISSN: 1309-3452

Let X be a linear space and let $(Z, \mu', \vartheta', \omega')$ be NNS. Let $\varphi : X \times X \times X \times X \to Z$ be a mapping such that for some $\alpha > 4$,
$\mu'\left(\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right), t\right) \geq \mu'(\varphi(x, y, z, w), \alpha t)$, $\vartheta'\left(\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right), t\right) \leq \vartheta'(\varphi(x, y, z, w), \alpha t)$ and
$\omega'\left(\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right), t\right) \leq \omega'(\varphi(x, y, z, w), \alpha t)$.
$\lim_{n\to\infty} \mu'\left(4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}\right), t\right) = 1$, $\lim_{n\to\infty} \vartheta'\left(4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}\right), t\right) = 0$ and
$\lim_{n\to\infty} \omega'\left(4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}\right), t\right) = 0$, for all $x, y, z, w \in X$ and all $t > 0$.
Let $(Y, \mu, \vartheta, \omega)$ be NBS and let $f: X \times X \to Y$ be a θ-approximately quadratic mapping in the sense of (3.1.2) with $f(0,0) = 0$. Then there exists a unique mapping $F: X \times X \to Y$ such that
$\mu(F(x, y) - f(x, y), t) \geq *^{\infty} \mu'(\varphi(x, x, y, y), (\alpha - 4)t)$,
$\vartheta(F(x, y) - f(x, y), t) \leq \diamondsuit^{\infty} \vartheta'(\varphi(x, x, y, y), (\alpha - 4)t)$ and
$\omega(F(x, y) - f(x, y), t) \leq \odot^{\infty} \omega'(\varphi(x, x, y, y), (\alpha - 4)t)$, for all $x, y, \in X$ and all $t > 0$.

Proof:

The techniques are similar to that of Theorem (3.1). Hence, we present a sketch of proof. Put $y = x$ and $w = z$ in (3.1.2) we get,
$\mu(f(2x, 2z) - 4f(x, z), t) \geq \mu'(\varphi(x, x, z, z,), t)$, $\vartheta(f(2x, 2z) - 4f(x, z), t) \leq \vartheta'(\varphi(x, x, z, z,), t)$ and
$\omega(f(2x, 2z) - 4f(x, z), t) \leq \omega'(\varphi(x, x, z, z,), t)$, for all $x, z, \in X$ and all $t > 0$. Thus we get,
$\mu\left(f(x, z) - 4f\left(\frac{x}{2}, \frac{z}{2}\right), t\right) \geq \mu'(\varphi(x, x, z, z), \alpha t)$,
$\vartheta\left(f(x, z) - 4f\left(\frac{x}{2}, \frac{z}{2}\right), t\right) \leq \vartheta'(\varphi(x, x, z, z), \alpha t)$ and
$\omega\left(f(x, z) - 4f\left(\frac{x}{2}, \frac{z}{2}\right), t\right) \leq \omega'(\varphi(x, x, z, z), \alpha t)$, for all $x, z, \in X$ and all $t > 0$.
For all $x, z, \in X$, all $m \geq 0$ all $n \in \mathbb{N}$ and all $t > 0$. We can deduce,

$$\left\{\begin{array}{l} \mu\left(4^m f\left(\frac{x}{2^m}, \frac{z}{2^m}\right) - 4^{n+m} f\left(\frac{x}{2^{n+m}}, \frac{z}{2^{n+m}}\right), t\right) \geq *^n \mu'\left(\varphi(x, x, z, z), \frac{t}{\sum_{k=m}^{n+m-1} \frac{4^k}{\alpha^{k+1}}}\right) \\ \vartheta\left(4^m f\left(\frac{x}{2^m}, \frac{z}{2^m}\right) - 4^{n+m} f\left(\frac{x}{2^{n+m}}, \frac{z}{2^{n+m}}\right), t\right) \leq \diamondsuit^n \vartheta'\left(\varphi(x, x, z, z), \frac{t}{\sum_{k=m}^{n+m-1} \frac{4^k}{\alpha^{k+1}}}\right) \text{ and} \\ \omega\left(4^m f\left(\frac{x}{2^m}, \frac{z}{2^m}\right) - 4^{n+m} f\left(\frac{x}{2^{n+m}}, \frac{z}{2^{n+m}}\right), t\right) \leq \odot^n \omega'\left(\varphi(x, x, z, z), \frac{t}{\sum_{k=m}^{n+m-1} \frac{4^k}{\propto^{k+1}}}\right). \end{array}\right\} \quad (3.3.1)$$

Hence the sequence $\left(4^n f\left(\frac{x}{2^n}, \frac{z}{2^n}\right)\right)$ is a Cauchy sequence in the NBS Y. Therefore there is a mapping $F: X \times X \to Y$ defined by $F(x, y) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)$, for all $x, y, \in X$.
The system of inequalities (3.3.1) with $m = 0$, implies that
$\mu(F(x, z) - f(x, z), t) \geq *^{\infty} \mu'(\varphi(x, x, z, z), (\alpha - 4)t)$,
$\vartheta(F(x, z) - f(x, z), t) \leq \diamondsuit^{\infty} \vartheta'(\varphi(x, x, z, z), (\alpha - 4)t)$ and
$\omega(F(x, z) - f(x, z), t) \leq \odot^{\infty} \omega'(\varphi(x, x, z, z), (\alpha - 4)t)$, for all $x, z, \in X$ and all $t > 0$.

4. Neutrosophic Continuity

Definition 4.1:

Let $g: \mathbb{R} \to X$ be a mapping, where $\mathbb{R}$ is endowed with the Euclidean topology and X is NNS equipped with Neutrosophic Norm (μ, ϑ, ω). Then $L \epsilon X$ is said to be Neutrosophic Limit of g at some $r_0 \in \mathbb{R}$ if and only if for every $\varepsilon > 0$ and $\alpha, \beta \in (0,1)$ there exists some $\delta = \delta(\varepsilon, \alpha, \beta) > 0$ such that
$\mu(g(r) - L, \varepsilon) \geq \alpha$ and $\mu(g(r) - L, \varepsilon) \leq 1 - \beta$ whenever $0 < |r - r_0| < \delta$.
In this case, we write $\lim_{r\to r_0} g(r) = L$ which also means that
$\lim_{r\to r_0} \mu(g(r) - L, t) = 0$, $\lim_{r\to r_0} \vartheta(g(r) - L, t) = 1$ and $\lim_{r\to r_0} \omega(g(r) - L, t) = 1$.
(or) $\mu(g(r) - L, t) = 1, \vartheta(g(r) - L, t) = 0$ and $\omega(g(r) - L, t) = 0$ as $r \to r_0$, for all $t > 0$.
The mapping g is said to be Neutrosophic Continuous at a point $r_0 \in \mathbb{R}$ if and only if
$\lim_{r\to r_0} \mu(g(r) - g(r_0), t) = 1$, $\lim_{r\to r_0} \vartheta(g(r) - g(r_0), t) = 0$ and

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https://publishoa.com
ISSN: 1309-3452

$\lim_{r\to r_0} \omega(g(r)-g(r_0),t)=0$ for all $t>0$.

Theorem 4.2:

Let X be a normed space and (Y,μ,ϑ,ω) be NBS. Let $(Z,\mu',\vartheta',\omega')$ be NNS, let $0<p<2$ and $z_0\in Z$. Let $f: X\times X\to Y$ be a mapping such that

$$\left\{\begin{array}{c}\mu(f(x+y,z+w)+f(x-y,z-w)-2f(x,z)-2f\ (y,w),t)\\ \geq \mu'\big((\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)z_0,t\big),\\ \\ \vartheta(f(x+y,z+w)+f(x-y,z-w)-2f(x,z)-2f\ (y,w),t)\\ \leq \vartheta'\big((\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)z_0,t\big) \text{ and}\\ \\ \omega(f(x+y,z+w)+f(x-y,z-w)-2f(x,z)-2f\ (y,w),t)\\ \leq \omega'\big((\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)z_0,t\big),\end{array}\right\} \quad (4.2.1)$$

for all $x,y,z,w\in X$ and all $t>0$. Assume that μ',ϑ' and ω'satisfies

$\lim_{n\to\infty}\mu'(2^{np}(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)z_0,4^n t)=1$,

$\lim_{n\to\infty}\vartheta'(2^{np}(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)z_0,4^n t)=0$ and

$\lim_{n\to\infty}\omega'(2^{np}(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)z_0,4^n t)=0$, for all $x,y,z,w\in X$ and all $t>0$.

Then there exists a unique mapping $F: X\times X\to Y$ Satisfying (3.1) such that

$$\left\{\begin{array}{c}\mu(F(x,y)-f(x,y),t)\geq *^\infty \mu'(2(\|x\|^p+\|y\|^p)z_0,(4-2^p)t),\\ \vartheta(F(x,y)-f(x,y),t)\leq \diamond^\infty \vartheta'(2(\|x\|^p+\|y\|^p)z_0,(4-2^p)t) \text{ and}\\ \omega(F(x,y)-f(x,y),t)\leq \odot^\infty \omega'(2(\|x\|^p+\|y\|^p)z_0,(4-2^p)t),\end{array}\right\} \quad (4.2.2)$$

for all $x,y\in X$ and all $t>0$. Furthermore if the mapping $g:\mathbb{R}\to Y$ defined by $g(r)=\frac{f(2^n rx,2^n ry)}{4^n}$ is Neutrosophic continuous for some $x,y\in X$ and all $n\in\mathbb{N}$ then the mapping $r\to F(rx,ry)$ from $\mathbb{R}$ to Y is Neutrosophic continuous. In this case $F(rx,ry)=r^2F(x,y)$, for all $r\in\mathbb{R}$.

Proof:

Define $\varphi: X\times X\times X\times X\to Z$ by $\varphi(x,y,z,w)=(\|x\|^p+\|y\|^P+\|z\|^P+\|w\|^P)z_0$, for all $x,y,z,w\in X$. Existence and uniqueness of the mapping F satisfying (3.1) and (4.2.2) are deduced from Theorem (3.1). Note that, for all $x,y,z,w\in X$, all $n\in\mathbb{N}$ and all $t>0$ we have,

$$\left\{\begin{array}{c}\mu\left(F(x,y)-\frac{f(2^n x,2^n y)}{4^n},t\right)=\mu\left(\frac{F(2^n x,2^n y)}{4^n}-\frac{f(2^n x,2^n y)}{4^n},t\right)\\ =\mu(F(2^n x,2^n y)-f(2^n x,2^n y),4^n t)\geq *^\infty \mu'(2^{np+1}(\|x\|^p+\|y\|^p)z_0,4^n(4-2^p)t),\\ \\ \vartheta\left(F(x,y)-\frac{f(2^n x,2^n y)}{4^n},t\right)=\vartheta\left(\frac{F(2^n x,2^n y)}{4^n}-\frac{f(2^n x,2^n y)}{4^n},t\right)\\ =\vartheta(F(2^n x,2^n y)-f(2^n x,2^n y),4^n t)\leq \diamond^\infty \vartheta'(2^{np+1}(\|x\|^p+\|y\|^p)z_0,4^n(4-2^p)t) \text{ and}\\ \\ \omega\left(F(x,y)-\frac{f(2^n x,2^n y)}{4^n},t\right)=\omega\left(\frac{F(2^n x,2^n y)}{4^n}-\frac{f(2^n x,2^n y)}{4^n},t\right)\\ \omega(F(2^n x,2^n y)-f(2^n x,2^n y),4^n t)\leq \odot^\infty \omega'(2^{np+1}(\|x\|^p+\|y\|^p)z_0,4^n(4-2^p)t).\end{array}\right\} \quad (4.2.3)$$

Putting $x=y=0$ in (3.1) we get,

$\mu\left(F(0,0)-\frac{1}{4^n}f(0,0)\right)\geq 1$, $\vartheta\left(F(0,0)-\frac{1}{4^n}f(0,0)\right)\leq 0$ and $\omega\left(F(0,0)-\frac{1}{4^n}f(0,0)\right)\leq 0$,

for all $n\in\mathbb{N}$ and all $t>0$. Fix $x,y\in X$ from (4.2.3) we get,

$$\mu\left(F(rx,ry)-\frac{f(2^n rx,2^n ry)}{4^n},t\right)\geq *^\infty \mu'\left((\|x\|^p+\|y\|^p)z_0,\frac{4^n(4-2^p)t}{2^{np+1}|r|^p}\right),$$

$$\vartheta\left(F(rx,ry)-\frac{f(2^n rx,2^n ry)}{4^n},t\right)\leq \diamond^\infty \vartheta'\left((\|x\|^p+\|y\|^p)z_0,\frac{4^n(4-2^p)t}{2^{np+1}|r|^p}\right) \text{ and}$$

$$\omega\left(F(rx,ry)-\frac{f(2^n rx,2^n ry)}{4^n},t\right)\leq \odot^\infty \omega'\left((\|x\|^p+\|y\|^p)z_0,\frac{4^n(4-2^p)t}{2^{np+1}|r|^p}\right),$$

for all $r\in\mathbb{R}\setminus\{0\}$. Since $\lim_{n\to\infty}\frac{4^n(4-2^p)t}{2^{np+1}|r|^p}=\infty$, for all $t>0$ we get,

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1343-1353
https://publishoa.com
ISSN: 1309-3452

$\lim_{n\to\infty}\mu\left(F(rx,ry)-\frac{f(2^n rx,2^n ry)}{4^n},\frac{t}{3}\right)=1, \lim_{n\to\infty}\vartheta\left(F(rx,ry)-\frac{f(2^n rx,2^n ry)}{4^n},\frac{t}{3}\right)=0$ and $\lim_{n\to\infty}\omega\left(F(rx,ry)-\frac{f(2^n rx,2^n ry)}{4^n},\frac{t}{3}\right)=0$, for all $r\in\mathbb{R}\setminus\{0\}$. Fix $r_0\in\mathbb{R}$.

By the Neutrosophic Continuity of the mapping $t\to\frac{f(2^n tx,2^n ty)}{4^n}$, we have,

$\lim_{n\to\infty}\mu\left(\frac{f(2^n rx,2^n ry)}{4^n}-\frac{f(2^n r_0x,2^n r_0y)}{4^{n_0}},\frac{t}{3}\right)=1, \lim_{n\to\infty}\vartheta\left(\frac{f(2^n rx,2^n ry)}{4^n}-\frac{f(2^n r_0x,2^n r_0y)}{4^{n_0}},\frac{t}{3}\right)=0$ and $\lim_{n\to\infty}\omega\left(\frac{f(2^n rx,2^n ry)}{4^n}-\frac{f(2^n r_0x,2^n r_0y)}{4^{n_0}},\frac{t}{3}\right)=0$. It follows that,

$$\mu(F(rx,ry)-F(r_0x,r_0y),t)\geq\begin{bmatrix}\mu\left(F(rx,ry)-\frac{f(2^n rx,2^n ry)}{4^n},\frac{t}{3}\right)\\ *\,\mu\left(\frac{f(2^n rx,2^n ry)}{4^n}-\frac{f(2^n r_0x,2^n r_0y)}{4^n},\frac{t}{3}\right)\\ *\,\mu\left(\frac{f(2^n r_0x,2^n r_0y)}{4^n}-F(r_0x,r_0y),\frac{t}{3}\right)\end{bmatrix}\geq 1,$$

$$\vartheta(F(rx,ry)-F(r_0x,r_0y),t)\leq\begin{bmatrix}\vartheta\left(F(rx,ry)-\frac{f(2^n rx,2^n ry)}{4^n},\frac{t}{3}\right)\\ \diamond\,\vartheta\left(\frac{f(2^n rx,2^n ry)}{4^n}-\frac{f(2^n r_0x,2^n r_0y)}{4^n},\frac{t}{3}\right)\\ \diamond\,\vartheta\left(\frac{f(2^n r_0x,2^n r_0y)}{4^n}-F(r_0x,r_0y),\frac{t}{3}\right)\end{bmatrix}\leq 0 \text{ and}$$

$$\omega(F(rx,ry)-F(r_0x,r_0y),t)\leq\begin{bmatrix}\omega\left(F(rx,ry)-\frac{f(2^n rx,2^n ry)}{4^n},\frac{t}{3}\right)\\ \odot\,\omega\left(\frac{f(2^n rx,2^n ry)}{4^n}-\frac{f(2^n r_0x,2^n r_0y)}{4^n},\frac{t}{3}\right)\\ \odot\,\omega\left(\frac{f(2^n r_0x,2^n r_0y)}{4^n}-F(r_0x,r_0y),\frac{t}{3}\right)\end{bmatrix}\leq 0,$$

as $r\to r_0$, for all $t>0$. Hence, the mapping $r\to F(rx,ry)$ is Neutrosophic Continuous. Now, we use the Neutrosophic Continuity of the mapping $r\to F(rx,ry)$ to establish that $F(sx,sy)=s^2F(x,y)$, for all $s\in\mathbb{R}$.

Fix $s\in\mathbb{R}$ and $t>0$. Then for each $\alpha\in\mathbb{R}$ with $0<\alpha<1$, there exists $\delta>0$ such that

$\mu\left(F(rx,ry)-F(sx,sy),\frac{t}{3}\right)\geq\alpha$, $\vartheta\left(F(rx,ry)-F(sx,sy),\frac{t}{3}\right)\leq 1-\alpha$ and $\omega\left(F(rx,ry)-F(sx,sy),\frac{t}{3}\right)\leq 1-\alpha$.

Choose a rational number r with $0<|r-s|<\delta$ and $0<|r^2-s^2|<1-\alpha$. Then

$$\mu(F(sx,sy)-s^2F(x,y),t)\geq\begin{bmatrix}\mu\left(F(sx,sy)-F(rx,ry),\frac{t}{3}\right)\\ *\,\mu\left(F(rx,ry)-r^2F(x,y),\frac{t}{3}\right)\\ *\,\mu\left(r^2F(x,y)-s^2F(x,y),\frac{t}{3}\right)\end{bmatrix}\geq\alpha*1*\mu\left(F(x,y),\frac{t}{3(1-\alpha)}\right),$$

$$\vartheta(F(sx,sy)-s^2F(x,y),t)\leq\begin{bmatrix}\vartheta\left(F(sx,sy)-F(rx,ry),\frac{t}{3}\right)\\ \diamond\,\vartheta\left(F(rx,ry)-r^2F(x,y),\frac{t}{3}\right)\\ \diamond\,\vartheta\left(r^2F(x,y)-s^2F(x,y),\frac{t}{3}\right)\end{bmatrix}\leq(1-\alpha)\diamond 0\diamond\vartheta\left(F(x,y),\frac{t}{3(1-\alpha)}\right) \text{ and}$$

$$\omega(F(sx,sy)-s^2F(x,y),t)\leq\begin{bmatrix}\omega\left(F(sx,sy)-F(rx,ry),\frac{t}{3}\right)\\ \odot\,\omega\left(F(rx,ry)-r^2F(x,y),\frac{t}{3}\right)\\ \odot\,\omega\left(r^2F(x,y)-s^2F(x,y),\frac{t}{3}\right)\end{bmatrix}\leq(1-\alpha)\odot 0\odot\omega\left(F(x,y),\frac{t}{3(1-\alpha)}\right).$$

Letting $\alpha\to 1$ and using the definition of NNS we get,

$\mu(F(sx,sy)-s^2F(x,y),t)=1, \vartheta(F(sx,sy)-s^2F(x,y),t)=0$ and $\omega(F(sx,sy)-s^2F(x,y),t)=0$.

Hence we conclude that $F(sx,sy)=s^2F(x,y)$.

Theorem 4.3:

Let X be a normed space and (Y,μ,ϑ,ω) be NBS. Let $(Z,\mu',\vartheta',\omega')$ be NNS and let $p>2$ and $z_0\in Z$. Let $f:X\times X\to Y$ be a mapping such that (4.2.1) for all $x,y,z,w\in X$ and let $t>0$. Assume that μ',ϑ' and ω' satisfies

$\lim_{n\to\infty}\mu'(2^{np}(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)z_0,4^n t)=1$,

$\lim_{n\to\infty}\vartheta'(2^{np}(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)z_0,4^n t)=0$ and

$\lim_{n\to\infty} \omega'(2^{np}(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, 4^n t) = 0,$
for all $x, y, z, w \in X$ and all $t > 0$. Then there exists a unique mapping $F: X \times X \to Y$ satisfying (3.1) such that

$$\left\{\begin{array}{c} \mu(F(x,y) - f(x,y), t) \geq *^{\infty} \mu'(2(\|x\|^p + \|y\|^p)z_0, (4-2^p)t), \\ \vartheta(F(x,y) - f(x,y), t) \leq \diamond^{\infty} \vartheta'(2(\|x\|^p + \|y\|^p)z_0, (4-2^p)t) \text{ and} \\ \omega(F(x,y) - f(x,y), t) \leq \odot^{\infty} \omega'(2(\|x\|^p + \|y\|^p)z_0, (4-2^p)t), \end{array}\right\} (4.3.1)$$

for all $x, y \in X$ and all $t > 0$. Furthermore if the mapping $g: \mathbb{R} \to Y$ defined by $g(r) = f(2^n rx, 2^n ry)$ is Neutrosophic Continuous then the mapping $r \to F(rx, ry)$ from $\mathbb{R}$ to Y is Neutrosophic Continuous.
In this case, $F(rx, ry) = r^2 F(x, y)$, for all $r \in \mathbb{R}$.

Proof:

Define $\varphi: X \times X \times X \times X \to Z$ by $\varphi(x,y,z,w) = (\|x\|^p + \|y\|^P + \|z\|^P + \|w\|^P)z_0$, for all $x, y, z, w \in X$. Then $\mu'\left(\varphi\left(\frac{x}{2},\frac{x}{2},\frac{y}{2},\frac{y}{2}\right),t\right) = \mu'\left(\frac{1}{2^{p-1}}(\|x\|^p + \|y\|^p)z_0, t\right)$, $\vartheta'\left(\varphi\left(\frac{x}{2},\frac{x}{2},\frac{y}{2},\frac{y}{2}\right),t\right) = \vartheta'\left(\frac{1}{2^{p-1}}(\|x\|^p + \|y\|^p)z_0, t\right)$ and $\omega'\left(\varphi\left(\frac{x}{2},\frac{x}{2},\frac{y}{2},\frac{y}{2}\right),t\right) = \omega'\left(\frac{1}{2^{p-1}}(\|x\|^p + \|y\|^p)z_0, t\right)$ for all $x, y \in X$ and all $t > 0$. Since $p > 2$, we have $2^p > 4$. By Theorem (3.3) ther exists a unique mapping F satisfying (3.1) and (4.3.1). The rest of the proof can be done on the same lines as in Theorem (4.2).

5. Conclusion

We linked here two different disciplines, namely, the fuzzy spaces and functional equations. We established Hyers–Ulam–Rassias stability of a functional equation $f(x+y, z+w) + f(x-y, z-w) = 2f(x,z) + 2f(y,w)$ in NNS.

We also studied Neutrosophic Continuity of the 2-dimensional vector variable quadratic mapping and the existence of a solution for any approximately 2-dimensional vector variable quadratic mapping of NNS.

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JOURNAL OF ALGEBRAIC STATISTICS
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