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On Generalized Cesaro Summability Method In Neutrosophic Normed Spaces Using Two- Sided Taubarian Conditions

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ABSTRACT:
We define the concept of Cesaro summability method in Neutrosophic normed spaces and prove a related Tauberian theorem. Also, we define slowly oscillating sequences in Neutrosophic normed spaces, prove related theorems and that Cesaro summability of slowly oscillating sequences implies ordinary convergence in Neutrosophic normed spaces. Finally, we give an analogue of classical two - sided Tauberian theorem.
Keywords: Neutrosophic normed spaces, Cesaro Summability, Tauberian theorem, Slow oscillation.
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1. Introduction

The theory of fuzzy sets was generalized from classical sets by [12] Zadeh in 1965. Which was further generalized to intuitionstic fuzzy sets by Atanassov [2]. This theory deals with a situation that may be imprecise or vague or uncertain by attributing a degree of membership and a degree of non-membership to a certain object several literature work on their corresponding sequence space can be found in 2004.

The idea of neutrosophic sets was introduced by [11] Smarandache as an extension of the intuitionstic fuzzy set. For the situation when the aggregate of the components is 1, in the wake of satisfying the condition by applying the Neutrosophic set operators, different outcomes can be acquired by applying the intuitionistic fuzzy operators, since the operators disregard the indeterminacy, while the neutrosophic operators are taked into the cognizance of the indeterminacy at similar level as truth membership and falsehood non-membership. Using the idea of neutrosophic sets, the notion of neutrosophic bipolar vague soft set and its application to decision making problems were defined. Summability theory and matrix transformation have been necessary modes in developing the theory and matrix transformation have been necessary modes in developing the theory of non-converging sequences. Recently, Jeyaraman et al. [4] introduced the notion of Hyers-Ulam-Rassias stability for functional equation in neutrosophic normed spaces.

The aim of this paper is to provide researchers with an introduction to summability theory and Taubarian theory for Neutrosophic normed spaces, in order to handle sequences which fail to converges ordinarily in the Neutrosophic normed spaces and to recover the convergence. We defined Cesaro Summability method in Neutrosophic normed spaces and give two- sided Taubarian conditions under which Cesaro Summability implies ordinary convergence in the space. Next, we define the concept of slowly oscillating sequence for neutrosophic normed space a concept which plays a vitual role in the development of classical Tauberian theory and compare the concept in neutrosophic normed space with that in classical normed spaces. Finally, we also give Tauberian conditions of slowly oscillating type and of Hardy's two- sided type by means of the concept of slowly oscillating sequence and q-boundedness in Neutrosophic normed spaces.

2. Preliminaries

In this section, we give some preliminaries for Neutrosophic Normed Spaces [NNS].

Definition 2.1:

The (V, μ, ν, ω) is said to be an NNS if V is a real vector space and $\mu, \nu \; and \; \omega$ are fuzzy sets on $V \times \mathbb{R}$ Satisfying the following conditions: For every $x, y \in V$ and $s, t \in \mathbb{R}$.

a) $0 \leq \mu(x,t) \leq 1$, $0 \leq \nu(x,t) \leq 1$, $0 \leq \omega(x,t) \leq 1 \; for \; all \; t \in R^+$,
b) $\mu(x,t) + \nu(x,t) + \omega(x,t) \geq 3 \; for \; t \in R^+$,
c) $\mu(x,t) = 0$ for all non-positive real number t,
d) $\mu(x,t) = 1$ for all $t \in R^+$ if and only if $x = 0$,
e) $\mu(cx,t) = \mu\left(x, \frac{t}{|c|}\right)$ for all $t \in R^+$ and $c \neq 0$,
f) $\mu(x+y, t+s) \geq min\{\mu(x,t), \mu(y,s)\}$,
g) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$,

h) $\nu(x,t)=1$ for all non-positive real number t,
i) $\nu(x,t)=0$ for all $t\in R^{+}$ if and only if $x=0$,
j) $\nu(cx,t)=\nu\left(x,\frac{t}{|c|}\right)$ for all $t\in R^{+}$ and $c\neq 0$,
k) $\nu(x+y,t+s)\leq max\{\nu(x,t),\nu(y,s)\}$,
l) $\lim_{t\to\infty}\nu(x,t)=0$ and $\lim_{t\to 0}\nu(x,t)=1$,
m) $\omega(x,t)=1$ for all non-positive real number t,
n) $\omega(x,t)=0$ for all $t\in R^{+}$ if and only if $x=0$,
o) $\omega(cx,t)=\omega\left(x,\frac{t}{|c|}\right)$ for all $t\in R^{+}$ and $c\neq 0$,
p) $\omega(x+y,t+s)\leq \max\{\omega(x,t),\omega(y,s)\}$,
q) $\lim_{t\to\infty}\omega(x,t)=0$ and $\lim_{t\to 0}\omega(x,t)=1$.

In this case, we will call (μ,ν,ω) an NNS on V. It is easy to see that for every $x\in V$ the functions $\mu(x,.)$, $\nu(x,.)$ and $\omega(x,.)$ are non-decreasing, non-increasing and non-increasing on $\mathbb{R}$ respectively.

Example 2.2:

Let $(V,\|\cdot\|)$ be a normed space μ_0,ν_0 *and* $\omega_0$and $V\times\mathbb{R}$ be F-sets on defined by

$$\mu_0(x,t)=\begin{Bmatrix}0 & t\leq 0\\ \frac{t}{t+\|x\|} & t>0\end{Bmatrix},\quad \nu_0(x,t)=\begin{Bmatrix}0 & t\leq 0\\ \frac{\|x\|}{t+\|x\|} & t>0\end{Bmatrix}\ \text{and}$$

$$\omega_0(x,t)=\begin{Bmatrix}0 & t\leq 0\\ \frac{\|x\|}{t} & t>0\end{Bmatrix}.$$ Then (μ_0,ν_0,ω_0) is NN on V.

Theorem 2.3:

Let (V,μ,ν,ω) be a normed space. Assume further that $\mu(x,t)>0$ for all $t>0$ implies $x=0$.

Define $\|x\|_\alpha=\inf\{t>0:\mu(x,t)>\alpha,\ \nu(x,t)<1-\alpha\ and\ \omega(x,t)<1-\alpha\}$, where $\alpha\in(0,1)$.
Then $\{\|x\|_\alpha:\alpha\in(0,1)\}$ is an ascending family of norms on V.

We note that these norms are called α-norms on V corresponding to (or including by) the Neutrosophic norm $(\mu,\ \nu,\ \omega)$ on V.

Next, we give some concepts concerning sequences in a NNS.

Definition 2.4:

A sequence(x_n) is said to be convergent to $x\in V$ in the NNS (V,μ,ν,ω) and denoted by $x_n\to x$, if for each $t>0$ and each $\varepsilon\in(0,1)$ there exists $n_0\in\mathbb{N}$ such that
$\mu(x_n-x,t)>1-\varepsilon$, $\nu(x_n-x,t)<\varepsilon$, and $\omega(x_n-x,t)<\varepsilon$ for all $n\geq n_0$.

Definition 2.5:

A sequence (x_n) in an NNS (V,μ,ν,ω) is said to be Cauchy if for each $t>0$ and each $\varepsilon\in(0,1)$ there exists $n_0\in\mathbb{N}$ such that $\mu(x_k-x_n,t)>1-\varepsilon,\nu(x_k-x_n,t)<\varepsilon$, and $\omega(x_k-x_n,t)<\varepsilon$, for all $k,n\geq n_0$.

Definition 2.6:

A sequence (x_n) in an NNS (V,μ,ν,ω) is G-Cauchy if $\lim_{n\to\infty}\mu\left(x_{n+p}-x_n,t\right)=1$, $\lim_{n\to\infty}\nu\left(x_{n+p}-x_n,t\right)=0$ and $\lim_{n\to\infty}\omega\left(x_{n+p}-x_n,t\right)=0$ for each $t>0$ and $p\in\mathbb{N}$.
This definition is equivalent to $\lim_{n\to\infty}\mu(x_{n+1}-x_n,t)=1$, $\lim_{n\to\infty}\nu(x_{n+1}-x_n,t)=0$ and $\lim_{n\to\infty}\omega(x_{n+1}-x_n,t)=0$, for each $t>0$.

We note that every convergent sequence is Cauchy and every Cauchy sequence is G-Cauchy in a NNS.

Next, we remind the concept of boundedness and q-boundedness in an NNS.

Definition 2.7:

Let (v,μ,ν,ω) be an NNS and A be a any subset of V.

(i) A is called bounded if there exist some $r\in(0,1)$ and $t_0>0$ such that $\mu(x,t_0)>1-r$, $\nu(x,t_0)<r$ and $\omega(x,t_0)<r$, for every $x\in A$.

(ii) A is said to be q-bounded if $\lim_{n\to\infty}\varphi_A(t)=1$, $\lim_{n\to\infty}\psi_A(t)=0$, and $\lim_{n\to\infty}€_A(t)=0$ where $\varphi_A(t)=inf\{\mu(x,t):x\in A\}$, $\psi_A(t)=sup\{\nu(x,t):x\in A\}$, and $€_A(t)=sup\{\nu(x,t):x\in A\}$.

It is obvious that a sequence NNS (μ,ν,ω) is bounded if and only if there exists some $t_0>0$ and $r\in(0,1)$ such that $\mu(x_n,t_0)>1-r$, $\nu(x_n,t_0)<r$ and $\omega(x_n,t_0)<r$ for every positive interger n and q-bounded if and only if $\lim_{t\to\infty}\inf_{n\in\mathbb{N}}\mu(x_n,1)=1$, $\lim_{t\to\infty}\inf_{n\in\mathbb{N}}\nu(x_n,1)=0$ and $\lim_{t\to\infty}\inf_{n\in\mathbb{N}}\omega(x_n,1)=0$.

3. Cesaro Summability Method In Neutrosophic Normed Spaces

Now, we define Cesaro Summability method in NNS and give related Taberian theorems.

Definition 3.1:

Let (x_n) be a sequence in an NNS (V, μ, ν, ω). The arithmetic means σ_n of (x_n) is defined by $\sigma_n = \frac{1}{n+1}(x+a)^n = \sum_{k=0}^{n} x_k$. We say that (x_n) is Cesaro summable to $\ell \in \mathbb{N}$ if $\lim_{n\to\infty} \sigma_n = \ell$.

First, we will show that Cesaro summability method is regular in an NNS.

Theorem 3.2:

Let (x_n) be a sequence in an NNS (V, μ, ν, ω). If (x_n) is convergent to $\ell \in V$, then (x_n) is cesaro summable to ℓ.

Proof:

Let (x_n) be a sequence in an NNS (V, μ, ν, ω) and (x_n) is convergent to $\ell \in V$. Fix $t > 0$. Then $\varepsilon > 0$

- There exists $n_0 \in \mathbb{N}$ such that $\mu\left(x_n - \ell, \frac{t}{2}\right) > 1 - \varepsilon$, $\nu\left(x_n - \ell, \frac{t}{2}\right) < \varepsilon$ and $\omega\left(x_n - \ell, \frac{t}{2}\right) < \varepsilon$ for $n > n_0$.
- There exist $n_1 \in \mathbb{N}$such that $\mu\left(\sum_{k=0}^{n_0}(x_k - \ell), \frac{(n+1)t}{2}\right) > 1 - \varepsilon$, $\nu\left(\sum_{k=0}^{n_0}(x_k - \ell), \frac{(n+1)t}{2}\right) > \varepsilon$ *and* $\omega\left(\sum_{k=0}^{n_0}(x_k - \ell), \frac{(n+1)t}{2}\right) > \varepsilon$.

For $n > n_1$ in view of the facts that

$\lim_{n\to\infty} \mu\left(\sum_{k=0}^{n_0}(x_k - \ell), \frac{(n+1)t}{2}\right) = 1,$

$$\lim_{n\to\infty} \nu\left(\sum_{k=0}^{n_0}(x_k - \ell), \frac{(n+1)t}{2}\right) = 0 \text{ and } \lim_{n\to\infty} \omega\left(\sum_{k=0}^{n_0}(x_k - \ell), \frac{(n+1)t}{2}\right) = 0.$$

Then, we have

$$\mu\left(\frac{1}{n+1}\sum_{k=0}^{n} x_k - \ell, t\right) = \mu\left(\frac{1}{n+1}\sum_{k=0}^{n}(x_k - \ell), t\right) = \mu\left(\sum_{k=0}^{n}(x_k - \ell), (n+1)t\right)$$

$$\geq min\left\{\mu\left(\sum_{k=0}^{n_0}(x_k - \ell), \frac{(n+1)t}{2}\right), \mu\left(\sum_{k=0}^{n}(x_k - \ell), \frac{(n+1)t}{2}\right)\right\}$$

$$\geq min\left\{\mu\left(\sum_{k=0}^{n_0}(x_k - \ell), \frac{(n+1)t}{2}\right), \mu\left(\sum_{k=0}^{n}(x_k - \ell), \frac{(n-n_0)t}{2}\right)\right\}$$

$$\geq min\left\{\begin{matrix}\mu\left(\sum_{k=0}^{n_0}(x_k - \ell), \frac{(n+1)t}{2}\right), \mu\left(x_{n_0+1} - \ell, \frac{t}{2}\right), \\ \mu\left(x_{n_0+2} - \ell, \frac{t}{2}\right), \ldots \mu\left(x_n - \ell, \frac{t}{2}\right)\end{matrix}\right\}$$

$$> 1 - \varepsilon,$$

$$\nu\left(\frac{1}{n+1}\sum_{k=0}^{n} x_k - \ell, t\right) = \nu\left(\frac{1}{n+1}\sum_{k=0}^{n}(x_k - \ell), t\right) = \nu\left(\sum_{k=0}^{n}(x_k - \ell), (n+1)t\right)$$

$$< max\left\{\nu\left(\sum_{k=0}^{n_0}(x_k - \ell), \frac{(n+1)t}{2}\right), \nu\left(\sum_{k=0}^{n}(x_k - \ell), \frac{(n+1)t}{2}\right)\right\}$$

$$< max\left\{\nu\left(\sum_{k=0}^{n_0}(x_k - \ell), \frac{(n+1)t}{2}\right), \nu\left(\sum_{k=0}^{n}(x_k - \ell), \frac{(n-n_0)t}{2}\right)\right\}$$

$$\leq max\left\{\begin{matrix}\nu\left(\sum_{k=0}^{n_0}(x_k - \ell), \frac{(n+1)t}{2}\right), \nu\left(x_{n_0+1} - \ell, \frac{t}{2}\right), \\ \nu\left(x_{n_0+2} - \ell, \frac{t}{2}\right), \ldots \nu\left(x_n - \ell, \frac{t}{2}\right)\end{matrix}\right\} < \varepsilon \text{ } and$$

$$\omega\left(\frac{1}{n+1}\sum_{k=0}^{n} x_k - \ell, t\right) = \omega\left(\frac{1}{n+1}\sum_{k=0}^{n}(x_k - \ell), t\right) = \omega\left(\sum_{k=0}^{n}(x_k - \ell), (n+1)t\right)$$

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$$< max\left\{\omega\left(\sum_{k=0}^{n_0}(x_k-\ell),\frac{(n+1)t}{2}\right),\omega\left(\sum_{k=0}^{n}(x_k-\ell),\frac{(n+1)t}{2}\right)\right\}$$
$$< max\left\{\omega\left(\sum_{k=0}^{n_0}(x_k-\ell),\frac{(n+1)t}{2}\right),\omega\left(\sum_{k=0}^{n}(x_k-\ell),\frac{(n-n_0)t}{2}\right)\right\}$$
$$\le max\left\{\begin{matrix}\omega\left(\sum_{k=0}^{n_0}(x_k-\ell),\frac{(n+1)t}{2}\right),\omega\left(x_{n_0+1}-\ell,\frac{t}{2}\right),\\ \omega\left(x_{n_0+2}-\ell,\frac{t}{2}\right),\dots\omega\left(x_n-\ell,\frac{t}{2}\right)\end{matrix}\right\}<\varepsilon\,.$$

Whenever $n > max\{n_0, n_1\}$, which completes the proof.
However, Cesaro Summability of a sequence in NNS does imply convergence which can be seen by following example.
Example 3.3:
Consider sequence $(x_n) = ((-1)^{n+1})$ in NNS $(\mathbb{R}, \mu_0, \nu_0, \omega_0)$ where μ_0, ν_0 and ω_0 are as in Example (2.2). Then, since

$$\lim_{n\to\infty}\mu_0(\sigma_{2n},t)=\lim_{n\to\infty}\mu_0\left(-\frac{1}{2n+1},t\right)=\lim_{n\to\infty}\frac{t}{t+\left|-\frac{1}{2n+1}\right|}=1,$$
$$\lim_{n\to\infty}\nu_0(\sigma_{2n},t)=\lim_{n\to\infty}\nu_0\left(-\frac{1}{2n+1},t\right)=\lim_{n\to\infty}\frac{\left|-\frac{1}{2n+1}\right|}{t+\left|-\frac{1}{2n+1}\right|}=0\ \ and$$
$$\lim_{n\to\infty}\omega_0(\sigma_{2n},t)=\lim_{n\to\infty}\omega_0\left(-\frac{1}{2n+1},t\right)=\lim_{n\to\infty}\frac{\left|-\frac{1}{2n+1}\right|}{t}=0.$$

We have $\sigma_{2n} \to \infty$ and since,

$$\lim_{n\to\infty}\mu_0(\sigma_{2n+1},t)=\lim_{n\to\infty}\mu_0(0,t)=\lim_{n\to\infty}\frac{t}{t+0}=1,$$
$$\lim_{n\to\infty}\nu_0(\sigma_{2n},t)=\lim_{n\to\infty}\nu_0(0,t)=\lim_{n\to\infty}\frac{0}{t+0}=0\ \ and$$
$$\lim_{n\to\infty}\omega_0(\sigma_{2n},t)=\lim_{n\to\infty}\omega_0(0,t)=\lim_{n\to\infty}\frac{0}{t}=0.$$

We have $\sigma_{2n+1} \to 0$ which implies that $\lim_{n\to\infty}\sigma_n = 0.$. Hence sequence (x_n) is Cesaro Summable to 0. However, sequence (x_n) is not convergent since $x_{2n} \to -1$ and $x_{2n+1} \to 1$ in view of the facts that $\lim_{n\to\infty}\mu_0(x_{2n}-(-1),t) = \lim_{n\to\infty}\mu_0(-1-(-1),t) = 1,$

$$\lim_{n\to\infty}\nu_0(x_{2n}-(-1),t)=\lim_{n\to\infty}\nu_0(-1-(-1),t)=0\ \ and$$
$$\lim_{n\to\infty}\omega_0(x_{2n}-(-1),t)=\lim_{n\to\infty}\omega_0(-1-(-1),t)=0.$$

Let

$$\lim_{n\to\infty}\mu_0(x_{2n+1}-1,t)=\lim_{n\to\infty}\mu_0(1-1,t)=1,$$
$$\lim_{n\to\infty}\nu_0(x_{2n+1}-1,t)=\lim_{n\to\infty}\nu_0(1-1,t)=0\quad and$$
$$\lim_{n\to\infty}\omega_0(x_{2n+1}-1,t)=\lim_{n\to\infty}\omega_0(1-1,t)=0.$$

Form this point, we investigate conditions ensured that a sequence which is Cesaro summable is convergent. For this, we need the following lemmas.
Lemma 3.4:
Let us define $\langle\lambda\rangle$ for every $\lambda > 0$ by $\langle\lambda\rangle = \lambda - [\lambda]$.. then, the following statement hold:
(i) If $\lambda > 1$, then $\lambda_n > n$ for each $n \in \mathbb{N} \setminus \{0\}$ with $n \ge \langle\lambda\rangle^{-1}$.
(ii) If $0 < \lambda < 1$, then $\lambda_n < n$ for each $n \in \mathbb{N} \setminus \{0\}$ with $\lambda_n = [\lambda_n]$.
Lemma 3.5:
We have the following statements:

(i) Let $\lambda < 1$, for each $n \in \mathbb{N} \setminus \{0\}$ with $n \ge \frac{3\lambda-1}{\lambda(\lambda-1)}$, we have $\frac{\lambda}{\lambda-1} < \frac{\lambda_n+1}{\lambda_n-n} < \frac{2\lambda}{\lambda-1}$.
(ii) If $0 < \lambda < 1$, for each $n \in \mathbb{N} \setminus \{0\}$ with $n > \langle\lambda\rangle^{-1}$ we have $0 < \frac{\lambda_n+1}{\lambda_n-n} < \frac{2\lambda}{\lambda-1}$.
Theorem 3.6:
Let (x_n) be a sequence in an NNS (V, μ, ν, ω). If (x_n) is Cesaro summable to $\ell \in V$ then it convergence to ℓ if and only if for all $t > 0$.

$\sup_{\lambda>1} \lim_{n\to\infty} \mu\left(\frac{1}{\lambda_n - n}\sum_{k=n+1}^{n}(x_k - x_n), t\right) = 1,$ (3.6.1)
$\inf_{\lambda>1} \lim_{n\to\infty} \nu\left(\frac{1}{\lambda_n - n}\sum_{k=n+1}^{n}(x_k - x_n), t\right) = 0,$ (3.6.2)
$\inf_{\lambda>1} \lim_{n\to\infty} \omega\left(\frac{1}{\lambda_n - n}\sum_{k=n+1}^{n}(x_k - x_n), t\right) = 0.$ (3.6.3)

Proof:

Let (x_n) be a sequence in NNS (V, μ, ν, ω) and be Cesaro summable to $\ell \in V$.
Necessity, (x_n) convergesℓ. Fix $t > 0$. For any $\lambda > 1$ by lemma (3.4) for each $n \in \mathbb{N} \setminus \{0\}$ with $n \geq \langle\lambda\rangle^{-1}$.
We can write $x_k - \sigma_n = \frac{\lambda_n+1}{\lambda_n-n}\left(\sigma_{\lambda_n} - \sigma_n\right) - \frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n).$ (3.6.4)
By lemma (3.5) for $n \geq \frac{3\lambda-1}{\lambda(\lambda-1)}$, we have

$$\mu\left(\frac{\lambda_n+1}{\lambda_n-n}\left(\sigma_{\lambda_n} - \sigma_n\right), t\right) = \mu\left(\sigma_{\lambda_n} - \sigma_n, \frac{t}{\frac{\lambda_n+1}{\lambda_n-n}}\right) \geq \mu\left(\sigma_{\lambda_n} - \sigma_n, \frac{t}{\frac{2\lambda}{\lambda-1}}\right),$$

$$\nu\left(\frac{\lambda_n+1}{\lambda_n-n}\left(\sigma_{\lambda_n} - \sigma_n\right), t\right) = \nu\left(\sigma_{\lambda_n} - \sigma_n, \frac{t}{\frac{\lambda_n+1}{\lambda_n-n}}\right) \leq \nu\left(\sigma_{\lambda_n} - \sigma_n, \frac{t}{\frac{2\lambda}{\lambda-1}}\right) \quad and$$

$$\omega\left(\frac{\lambda_n+1}{\lambda_n-n}\left(\sigma_{\lambda_n} - \sigma_n\right), t\right) = \omega\left(\sigma_{\lambda_n} - \sigma_n, \frac{t}{\frac{\lambda_n+1}{\lambda_n-n}}\right) \leq \omega\left(\sigma_{\lambda_n} - \sigma_n, \frac{t}{\frac{2\lambda}{\lambda-1}}\right).$$

Since (σ_n) is Cauchy $\lim_{n\to\infty} \mu\left(\frac{\lambda_n+1}{\lambda_n-n}\left(\sigma_{\lambda_n} - \sigma_n\right), t\right) = 1$, $\lim_{n\to\infty} \nu\left(\frac{\lambda_n+1}{\lambda_n-n}\left(\sigma_{\lambda_n} - \sigma_n\right), t\right) = 0$
and $\lim_{n\to\infty} \omega\left(\frac{\lambda_n+1}{\lambda_n-n}\left(\sigma_{\lambda_n} - \sigma_n\right), t\right) = 0$. Which means that $\frac{\lambda_n+1}{\lambda_n-n}\left(\sigma_{\lambda_n} - \sigma_n\right) \to 0$.
So, by (3.6.4) $\lim_{n\to\infty} \mu\left(\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n), t\right) = 1,$
$\lim_{n\to\infty} \nu\left(\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n), t\right) = 0$ and $\lim_{n\to\infty} \omega\left(\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n), t\right) = 0.$
Consequently, (3.6.1), (3.6.2) and (3.6.3) are proved.
Sufficiency, suppose that (3.6.1), (3.6.2) and (3.6.3) are satisfied.
Fix $t > 0$ for given $\varepsilon > 0$, we have:

- There exist $\lambda > 1$ and $n_0 \in \mathbb{N}$ such that $\mu\left(\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n), \frac{t}{3}\right) > 1 - \varepsilon,$
  $\nu\left(\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n), \frac{t}{3}\right) < \varepsilon$ and $\omega\left(\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n), \frac{t}{3}\right) < \varepsilon$, for all $n > n_1$.
- There exists $n_1 \in \mathbb{N}$ such that $\mu\left(\sigma_n - \ell, \frac{t}{3}\right) > 1 - \varepsilon$, $\nu\left(\sigma_n - \ell, \frac{t}{3}\right) > 1 - \varepsilon$ and
  $\omega\left(\sigma_n - \ell, \frac{t}{3}\right) < \varepsilon$ for $n > n_1$.
- There exists $n_2 \in \mathbb{N}$ such that $\mu\left(\frac{\lambda_n+1}{\lambda_n-n}\left(\sigma_{\lambda_n} - \sigma_n\right), \frac{t}{3}\right) > 1 - \varepsilon$,
  $\nu\left(\frac{\lambda_n+1}{\lambda_n-n}\left(\sigma_{\lambda_n} - \sigma_n\right), \frac{t}{3}\right) < \varepsilon$ and $\omega\left(\frac{\lambda_n+1}{\lambda_n-n}\left(\sigma_{\lambda_n} - \sigma_n\right), \frac{t}{3}\right) < \varepsilon$, in view of the fact that
  $\frac{\lambda_n+1}{\lambda_n-n}\left(\sigma_{\lambda_n} - \sigma_n\right) \to 0$. Hence, we get

$$\begin{aligned}
\mu(x_n - \ell, t) &= \mu(x_n - \sigma_n + \sigma_n - \ell, t) \\
&= \mu\left(\frac{\lambda_n+1}{\lambda_n-n}\left(\sigma_{\lambda_n} - \sigma_n\right) - \frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n) + \sigma_n - \ell, t\right) \\
&\geq min\left\{\mu\left(\frac{\lambda_n+1}{\lambda_n-n}\left(\sigma_{\lambda_n} - \sigma_n\right), \frac{t}{3}\right), \mu\left(\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n), \frac{t}{3}\right), \mu\left(\sigma_n - \ell, \frac{t}{3}\right)\right\} \\
&> 1 - \varepsilon,
\end{aligned}$$

$$\begin{aligned}
\nu(x_n - \ell, t) &= \nu(x_n - \sigma_n + \sigma_n - \ell, t) \\
&= \nu\left(\frac{\lambda_n+1}{\lambda_n-n}\left(\sigma_{\lambda_n} - \sigma_n\right) - \frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n) + \sigma_n - \ell, t\right) \\
&\leq max\left\{\nu\left(\frac{\lambda_n+1}{\lambda_n-n}\left(\sigma_{\lambda_n} - \sigma_n\right), \frac{t}{3}\right), \nu\left(\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n), \frac{t}{3}\right), \nu\left(\sigma_n - \ell, \frac{t}{3}\right)\right\} \\
&< \varepsilon \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\omega(x_n - \ell, t) &= \omega(x_n - \sigma_n + \sigma_n - \ell, t) \\
&= \omega\left(\frac{\lambda_n+1}{\lambda_n-n}\left(\sigma_{\lambda_n} - \sigma_n\right) - \frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n) + \sigma_n - \ell, t\right)
\end{aligned}$$

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$$\leq max\left\{\omega\left(\frac{\lambda_n+1}{\lambda_n-n}(\sigma_{\lambda_n}-\sigma_n),\frac{t}{3}\right),\omega\left(\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(x_k-x_n),\frac{t}{3}\right),\omega\left(\sigma_n-\ell,\frac{t}{3}\right)\right\}$$
$$< \varepsilon.$$

For $n > max\{n_0, n_1, n_2\}$, which completes the proof.
In the case of $0 < \lambda < 1$ by using lemma (3.4) we have the following equality:

$$x_n - \sigma_n = \frac{\lambda_n + 1}{n - \lambda_n}(\sigma_n - \sigma_{\lambda_n}) + \frac{1}{n - \lambda_n}\sum_{k=\lambda_n+1}^{n}(x_k - x_k).$$

Using a similar argument in theorem (3.6) and equality (3.4), we give the following theorem. The proof is very similar to that of theorem (3.6) and hence, it is omitted.

Theorem 3.7:

Let (x_n) be a sequence in an NNS (V, μ, ν, ω). If (x_n) is Cesaro summable to $\ell \in V$, then it converges to ℓ if and only if for all $t > 0$.

$$\sup_{0<\lambda<1}\lim_{n\to\infty}\mu\left(\frac{1}{n-\lambda_n}\sum_{k=n+1}^{n}(x_n - x_k), t\right) = 1,$$
$$\inf_{0<\lambda<1}\lim_{n\to\infty}\nu\left(\frac{1}{n-\lambda_n}\sum_{k=n+1}^{n}(x_n - x_k), t\right) = 0 \quad \text{and}$$
$$\inf_{0<\lambda<1}\lim_{n\to\infty}\omega\left(\frac{1}{n-\lambda_n}\sum_{k=n+1}^{n}(x_n - x_k), t\right) = 0.$$

4 . Slowly Oscillating sequence in Neutrosophic Normed Spaces

We now introduce oscillating sequences in NNS and obtain related results.

Definition 4.1:

A sequence (x_n) in NNS (V, μ, ν, ω) is said to be slowly oscillating if

$$\sup_{0<\lambda<1}\liminf_{n\to\infty}\min_{n<k\leq\lambda_n}\mu(x_n - x_k, t) = 1, \qquad (4.1.1)$$
$$\inf_{0<\lambda<1}\liminf_{n\to\infty}\min_{n<k\leq\lambda_n}\nu(x_n - x_k, t) = 0, \qquad (4.1.2)$$
$$\inf_{0<\lambda<1}\liminf_{n\to\infty}\min_{n<k\leq\lambda_n}\omega(x_n - x_k, t) = 0. \qquad (4.1.3)$$

For all $t > 0$, where λ_n denotes the integer part of the product λ_n.
" $\underset{\lambda>1}{sup}$ " in (4.1.1), " $\underset{\lambda>1}{inf}$ " in (4.1.2) and " $\underset{\lambda>1}{inf}$ " in (4.1.3) can be replaced by " $\underset{\lambda\to 1^+}{lim}$ ".
Slow oscillation condition in an NNS can be rewritten as follows:
A sequence (x_n) is slowly oscillating if and only if for all $t > 0$ and for all $\varepsilon \in (0,1)$ there exist $\lambda > 1$ and $n_0 \in \mathbb{N}$, depending on t and ε such that $\mu(x_n - x_k, t) > 1 - \varepsilon$, $\nu(x_n - x_k, t) < \varepsilon$ *and* $\omega(x_n - x_k, t) < \varepsilon$, whenever $n_0 \leq n < k \leq \lambda_n$.

Theorem 4.2:

Let (x_n) be a sequence in NNS (V, μ, ν, ω). For each $t > 0$ conditions (4.1.1), (4.1.2), (4.1.3) of slowly oscillation are equivalent to

$$\sup_{0<\lambda<1}\liminf_{n\to\infty}\min_{\lambda_n<k\leq n}\mu(x_k - x_n, t) = 1, \qquad (4.2.4)$$
$$\inf_{0<\lambda<1}\limsup_{n\to\infty}\max_{\lambda_n<k\leq n}\nu(x_k - x_n, t) = 0 \qquad (4.2.5)$$
$$\inf_{0<\lambda<1}\limsup_{n\to\infty}\max_{\lambda_n<k\leq n}\omega(x_k - x_n, t) = 0 \qquad (4.2.6)$$

Respectively, " $\underset{0<\lambda<1}{\sup}$ " in (4.2.4), " $\underset{0<\lambda<1}{\inf}$ " in (4.2.5) and " $\underset{0<\lambda<1}{\inf}$ " in (4.2.6) can be replaced by " $\underset{\lambda\to 1^-}{\lim}$ ".

Proof:

We will show that (4.1.1) and (4.2.4) , (4.1.2) and (4.2.5) are equivalent, and equivalence of (4.1.3) and (4.2.6) can be done similarly. Fix $t > 0$.
Let $f(\lambda) = \liminf_{n\to\infty}\min_{n<k\leq\lambda_n}\mu(x_k - x_n, t)$ and

$$g\left(\frac{1}{\lambda}\right) = \liminf_{n\to\infty}\min_{\left[\frac{k}{\lambda}\right]<n\leq k}\mu(x_k - x_n, t).$$

Where $\lambda > 1$. Then, for any $\lambda > 1$ there exist an increasing sequence (n_p) such that

$$f(\lambda) = \lim_{p\to\infty}\min_{n_p<k\leq\lambda_{n_p}}\mu\left(x_k - x_{n_p}, t\right).$$

There also exists a sequence $k_p \in \left(n_p, \left[\lambda_{n_p}\right]\right)$ such that

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$\min_{n_p<k\le\lambda_{n_p}} \mu\left(x_k - x_{n_p}, t\right) = \mu\left(x_k - x_{n_p}, t\right)$.

Here, we note that $k_p \in \left(n_p, \left[\lambda_{n_p}\right]\right)$ implies $n_p \in \left(\left[\frac{k_p}{\lambda}\right], k_p\right)$.

Then, we get $g\left(\frac{1}{\lambda}\right) = \liminf_{k\to\infty} \min_{\left[\frac{k}{\lambda}\right]<n\le k} \mu(x_k - x_n, t)$

$$\le \lim_{p\to\infty} \min_{\left[\frac{k_p}{\lambda}\right]<n\le k} \mu\left(x_{k_p} - x_n, t\right)$$

$$\le \lim_{p\to\infty} \mu\left(x_{k_p} - x_{n_p}, t\right) = \lim_{p\to\infty} \min_{n_p<k\le\lambda_{n_p}} \mu\left(x_k - x_{n_p}, t\right) = f(\lambda).$$

Now, $f(\lambda) = \limsup_{n\to\infty} \max_{n<k\le\lambda_n} \nu(x_k - x_n, t)$ and

$g\left(\frac{1}{\lambda}\right) = \limsup_{n\to\infty} \max_{\left[\frac{k}{\lambda}\right]<n\le k} \nu(x_k - x_n, t)$. Where $\lambda > 1$. Then, for any $\lambda > 1$, there exist an decreasing sequence (n_p) such that $f(\lambda) = \lim_{p\to\infty} \max_{n_p<k\le\lambda_{n_p}} \nu\left(x_k - x_{n_p}, t\right)$.

There also exists a sequence $k_p \in \left(n_p, \left[\lambda_{n_p}\right]\right)$ such that

$\max_{n_p<k\le\lambda_{n_p}} \nu\left(x_k - x_{n_p}, t\right) = \nu\left(x_k - x_{n_p}, t\right)$.

Here, we note that $k_p \in \left(n_p, \left[\lambda_{n_p}\right]\right)$ implies $n_p \in \left(\left[\frac{k_p}{\lambda}\right], k_p\right)$. Then, we get

$$g\left(\frac{1}{\lambda}\right) = \limsup_{k\to\infty} \max_{\left[\frac{k}{\lambda}\right]<n\le k} \nu(x_k - x_n, t) \ge \lim_{p\to\infty} \max_{\left[\frac{k_p}{\lambda}\right]<n\le k} \nu\left(x_{k_p} - x_n, t\right)$$

$$\ge \lim_{p\to\infty} \nu\left(x_{k_p} - x_{n_p}, t\right) = \lim_{p\to\infty} \max_{n_p<k\le\lambda_{n_p}} \nu\left(x_k - x_{n_p}, t\right) = f(\lambda).$$

Now, $f(\lambda) = \limsup_{n\to\infty} \max_{n<k\le\lambda_n} \omega(x_k - x_n, t)$ and

$g\left(\frac{1}{\lambda}\right) = \limsup_{n\to\infty} \max_{\left[\frac{k}{\lambda}\right]<n\le k} \omega(x_k - x_n, t)$.

Where $\lambda > 1$. Then, for any $\lambda > 1$, there exist an deccreasing sequence (n_p) such that

$f(\lambda) = \lim_{p\to\infty} \max_{n_p<k\le\lambda_{n_p}} \omega\left(x_k - x_{n_p}, t\right)$.

There also exists a sequence $k_p \in \left(n_p, \left[\lambda_{n_p}\right]\right)$ such that

$\max_{n_p<k\le\lambda_{n_p}} \omega\left(x_k - x_{n_p}, t\right) = \omega\left(x_k - x_{n_p}, t\right)$.

Here, we note that $k_p \in \left(n_p, \left[\lambda_{n_p}\right]\right)$ implies $n_p \in \left(\left[\frac{k_p}{\lambda}\right], k_p\right)$. Then, we get

$$g\left(\frac{1}{\lambda}\right) = \limsup_{k\to\infty} \max_{\left[\frac{k}{\lambda}\right]<n\le k} \omega(x_k - x_n, t) \ge \lim_{p\to\infty} \max_{\left[\frac{k_p}{\lambda}\right]<n\le k} \omega\left(x_{k_p} - x_n, t\right)$$

$$\ge \lim_{p\to\infty} \omega\left(x_{k_p} - x_{n_p}, t\right) = \lim_{p\to\infty} \max_{n_p<k\le\lambda_{n_p}} \omega\left(x_k - x_{n_p}, t\right) = f(\lambda).$$

On the other hand, changing the roles of $f(\lambda)$ and $g\left(\frac{1}{\lambda}\right)$ applying the same procedure, we also get $g\left(\frac{1}{\lambda}\right) \ge f(\lambda)$. Hence, for any $\lambda > 1$, we have $f(\lambda) = g\left(\frac{1}{\lambda}\right)$, which implies equivalent of (4.1.1) and (4.2.4), (4.1.2) and (4.2.5) and (4.1.3) and (4.2.6).

Example 4.3:

Consider NNS $(\mathbb{R}, \mu_0, \nu_0, \omega_0)$ where μ_0, ν_0 and ω_0 are as in Example (2.2). Sequence $x_n = \sum_{j=1}^{n} \frac{1}{j}$ is slowly oscillating in $(\mathbb{R}, \mu_0, \nu_0, \omega_0)$ by the following:

Fix $t > 0$. Given $\varepsilon \in (0,1)$ choose $\lambda = \frac{t\varepsilon}{1-\varepsilon} + 1$. Then for $1 < n < k < \lambda_n$, we have

$$\mu_0(x_k - x_n, t) = \frac{t}{t + |x_k - x_n|} > \frac{t}{t + \frac{t\varepsilon}{1-\varepsilon}} = 1 - \varepsilon,$$

$$\nu_0(x_k - x_n, t) = \frac{|x_k - x_n|}{t + |x_k - x_n|} < \frac{\frac{t\varepsilon}{1-\varepsilon}}{t + \frac{t\varepsilon}{1-\varepsilon}} = \varepsilon \quad \text{and}$$

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$$\omega_0(x_k - x_n, t) = \frac{|x_k - x_n|}{t} < \frac{\frac{t\varepsilon}{1-\varepsilon}}{t} = \frac{\varepsilon}{1-\varepsilon} .$$

In view of the fact that $|x_k - x_n| = x_n = \sum_{j=n+1}^{n} \frac{1}{j} < \frac{k-n}{n} < \frac{k}{n} - 1 < \lambda - 1 = \frac{t\varepsilon}{1-\varepsilon}$, which implies (x_n) is slowly oscillating in NNS $(\mathbb{R}, \mu_0, \nu_0, \omega_0)$.

Theorem 4.4:

Let $(V, \|\cdot\|)$ be a normed space $(V, \mu_0, \nu_0, \omega_0)$ be NNS in example (2.2). A sequence (x_n) is slowly oscillating in $(V, \|\cdot\|)$ if and only if (x_n) is slowly oscillating in $(V, \mu_0, \nu_0, \omega_0)$.

Proof:

Let (x_n) be slowly oscillating in $(V, \|\cdot\|)$. Given $t > 0$ and $\varepsilon \in (0,1)$. We define $\varepsilon_0 = \frac{t\varepsilon}{1-\varepsilon} > 0$. Then there exists $\lambda > 1$ and $n_0 \in \mathbb{N}$ such that $\|x_k - x_n\| < \varepsilon_0$.
whenever $n_0 \le n < k \le \lambda_n$. So, $\mu_0(x_k - x_n, t) = \frac{t}{t + \|x_k - x_n\|} > \frac{t}{t+\varepsilon_0} = 1 - \varepsilon,$

$$\nu_0(x_k - x_n, t) = 1 - \mu_0(x_k - x_n, t) < \varepsilon , \nu_0(x_k - x_n, t) = \frac{1}{\nu_0(x_k - x_n, t)} - 1 < \frac{\varepsilon}{1-\varepsilon} .$$

Whenever $n_0 \le n < k \le \lambda_n$. This means that (x_n) is slowly oscillating in $(V, \mu_0, \nu_0, \omega_0)$.

Conversely, if (x_n) is slowly oscillating in $(V, \mu_0, \nu_0, \omega_0)$, given $\varepsilon \in (0,1/2)$ there exist $\lambda > 1$ and $n_0 \in \mathbb{N}$ such that $\mu_0(x_k - x_n, 1) = \frac{1}{1+ \|x_k - x_n\|} > 1 - \varepsilon$ whenever $n_0 \le n < k \le \lambda_n$.
So, we have $\|x_k - x_n\| < \frac{\varepsilon}{1-\varepsilon} < 2\varepsilon$.
Whenever $n_0 \le n < k \le \lambda_n$. We conclude that (x_n) is slowly oscillating in $(V, \|\cdot\|)$.
It is obvious that the following implication hold:
Cauchy ⇒ Slow oscillating ⇒ G-Cauchy.
Above implication cannot be reverted in general which can be seen by the following example.

Example 4.5:

Consider NNS $(\mathbb{R}, \mu_0, \nu_0, \omega_0)$ and μ_0, ν_0 *and* ω_0 are in example (2.2). By Theorem (4.4), the sequence (x_n) given by $x_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$ is G-Cauchy but is not slowly oscillating and the sequence (y_n) given by $y_n = \sum_{k=1}^{n} \frac{1}{k}$ is slowly oscillating but is not Cauchy.

Theorem 4.6:

Let (V, μ, ν, ω) be an NNS satisfying condition (2.1) and (x_n) be a sequence in V. (x_n) is slowly oscillating (V, μ, ν, ω) if and only if (x_n) is slowly oscillating in $(V, \|\cdot\|_\alpha)$ for each $\alpha \in (0,1)$.

Proof:

Let $\alpha \in (0,1)$ and $s > 0$ are given. Assume that (x_n) slowly oscillating (V, μ, ν, ω).
Then, for $\varepsilon = 1 - \alpha$ there exist $\lambda > 1$ and $n_0 \le n < k \le \lambda_n$ we obtain
$\mu_0(x_k - x_n, s) > 1 - \varepsilon$, $\nu_0(x_k - x_n, s) < \varepsilon$ and $\omega_0(x_k - x_n, s) < \varepsilon$.

$$\|x_k - x_n\|_\alpha = inf \begin{Bmatrix} t > 0: \mu(x_k - x_n, t) > \alpha,\ \nu(x_k - x_n, t) < 1 - \alpha \quad and \\ \omega(x_k - x_n, t) < 1 - \alpha \end{Bmatrix} < s$$

This means that (x_n) is slowly oscillating in $(V, \|\cdot\|_\alpha)$.

Conversely, choose $\alpha \in (0,1)$ and let (x_n) be a slowly oscillating sequence in $(V, \|\cdot\|_\alpha)$.
Then, for $s > 0$ there exist sequence in NNS (V, μ, ν, ω) and $n_0 \in \mathbb{N}$ such that

$$\|x_k - x_n\|_\alpha = inf \begin{Bmatrix} t > 0: \mu(x_k - x_n, t) > \alpha,\ \nu(x_k - x_n, t) < 1 - \alpha \quad and \\ \omega(x_k - x_n, t) < 1 - \alpha \end{Bmatrix} < s.$$

Whenever $n_0 \le n < k \le \lambda_n$. Thus $\mu(x_k - x_n, s) > \alpha$, $\nu(x_k - x_n , s) < 1 - \alpha$ and $\omega(x_k - x_n, s) < 1 - \alpha$. Whenever $n_0 \le n < k \le \lambda_n$. Since s and α were arbitrary, (x_n) is slowly oscillating (V, μ, ν, ω).

Theorem 4.7:

Let (x_n) be sequence in NNS (V, μ, ν, ω). If (x_n) is a slowly oscillating the (3.6.1), (3.6.2) and (3.6.3) are satisfied.

Proof:

Let (x_n) be sequence slowly oscillating sequence in NNS (V, μ, ν, ω). Fix $t > 0$. Then, for given $\varepsilon \in (0,1)$ there exist $\lambda > 1$ and $n_0 \in \mathbb{N}$ such that $\mu(x_k - x_n, t) > 1 - \varepsilon$, $\nu(x_k - x_n, t) < \varepsilon$ and $\omega(x_k - x_n, t) < \varepsilon$, whenever $n_0 \le n < k \le \lambda_n$.
Hence, we have

$$\mu\left(\frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} (x_k - x_n) , t\right) = \mu\left(\sum_{k=n+1}^{\lambda_n} (x_k - x_n) + (\lambda_n - n)t\right)$$
$$\ge min\{\mu(x_{n+1} - x_n, t), \mu(x_{n+2} - x_n, t), \dots, \mu(x_{\lambda_n} - x_n, t)\}$$

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$$> 1 - \varepsilon,$$

$$\nu\left(\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n), t\right) = \nu\left(\sum_{k=n+1}^{\lambda_n}(x_k - x_n) + (\lambda_n - n)t\right)$$

$$\leq max\{\nu(x_{n+1} - x_n, t), \nu(x_{n+2} - x_n, t), \dots, \nu(x_{\lambda_n} - x_n, t)\},$$

$$< \varepsilon \quad \text{and}$$

$$\omega\left(\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n), t\right) = \omega\left(\sum_{k=n+1}^{\lambda_n}(x_k - x_n) + (\lambda_n - n)t\right)$$

$$\leq max\{\omega(x_{n+1} - x_n, t), \omega(x_{n+2} - x_n, t), \dots, \omega(x_{\lambda_n} - x_n, t)\},$$

$$< \varepsilon.$$

whenever $n_0 \leq n < k \leq \lambda_n$, which completes the proof.
By theorem (3.6) and theorem (4.7), we can formulate the following result.

Theorem 4.8:

Let (x_n) be sequence in NNS (V, μ, ν, ω). If (x_n) is Cesaro Summable to $\ell \in V$ and slowly oscillating, then (x_n) converges to ℓ.

Next, we prove a comparison theorem between the concepts of slow oscillation and q-boundedness of sequence in NNS.

Theorem 4.9:

Let (x_n) be sequence in NNS (V, μ, ν, ω). If $\{n(x_n - x_{n-1})\}$ is q-bounded then (x_n) is slowly oscillating.

Proof:

Given $\varepsilon \in (0,1)$. By (2.2), there exist $M_\varepsilon > 0$ such that
$t > M_\varepsilon \Rightarrow \inf_{n\in\mathbb{N}} \mu(n(x_n - x_{n-1}), t) > 1 - \varepsilon$, $\sup_{n\in\mathbb{N}} \nu(n(x_n - x_{n-1}), t) < \varepsilon$ *and*
$\sup_{n\in\mathbb{N}} w(n(x_n - x_{n-1}), t) < \varepsilon$. For each $t > 0$,
if we take $\lambda < 1 + \frac{t}{M_\varepsilon}$, then $n_0 < n < k \leq \lambda_n$

$$\mu(x_k - x_n, t) = \mu\left(\sum_{j=n+1}^{k}(x_j - x_{j-1}), t\right) \geq \min_{n+1\leq j\leq k} \mu\left((x_j - x_{j-1}), \frac{t}{k-n}\right)$$

$$= \min_{n+1\leq j\leq k} \mu\left(j(x_j - x_{j-1}), \frac{jt}{k-n}\right) \geq \min_{n+1\leq j\leq k} \mu\left(j(x_j - x_{j-1}), \frac{nt}{k-n}\right)$$

$$\geq \min_{n+1\leq j\leq k} \mu\left(j(x_j - x_{j-1}), \frac{t}{\frac{k}{n} - n}\right) \geq \min_{n+1\leq j\leq k} \mu\left(j(x_j - x_{j-1}), \frac{t}{\lambda - 1}\right)$$

$$\geq \inf_{n\in\mathbb{N}} \mu\left(n(x_n - x_{n-1}), \tfrac{t}{\lambda-1}\right) > 1 - \varepsilon,$$

$$\nu(x_k - x_n, t) = \nu\left(\sum_{j=n+1}^{k}(x_j - x_{j-1}), t\right) \leq \max_{n+1\leq j\leq k} \mu\left((x_j - x_{j-1}), \frac{t}{k-n}\right)$$

$$= \max_{n+1\leq j\leq k} \nu\left(j(x_j - x_{j-1}), \frac{jt}{k-n}\right) \leq \max_{n+1\leq j\leq k} \nu\left(j(x_j - x_{j-1}), \frac{nt}{k-n}\right)$$

$$\leq \max_{n+1\leq j\leq k} \nu\left(j(x_j - x_{j-1}), \frac{t}{\frac{k}{n} - n}\right) \leq \max_{n+1\leq j\leq k} \nu\left(j(x_j - x_{j-1}), \frac{t}{\lambda - 1}\right)$$

$$\leq \sup_{n\in\mathbb{N}} \nu\left(n(x_n - x_{n-1}), \tfrac{t}{\lambda-1}\right) < \varepsilon \quad and$$

$$\omega(x_k - x_n, t) = \omega\left(\sum_{j=n+1}^{k}(x_j - x_{j-1}), t\right) \leq \max_{n+1\leq j\leq k} \omega\left((x_j - x_{j-1}), \frac{t}{k-n}\right)$$

$$= \max_{n+1\leq j\leq k} \omega\left(j(x_j - x_{j-1}), \frac{jt}{k-n}\right) \leq \max_{n+1\leq j\leq k} \omega\left(j(x_j - x_{j-1}), \frac{nt}{k-n}\right)$$

$$\leq \max_{n+1\leq j\leq k} \omega\left(j(x_j - x_{j-1}), \frac{t}{\frac{k}{n} - n}\right) \leq \max_{n+1\leq j\leq k} \omega\left(j(x_j - x_{j-1}), \frac{t}{\lambda - 1}\right)$$

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$$\leq \sup_{n\in\mathbb{N}} \omega\left(n(x_n - x_{n-1}), \tfrac{t}{\lambda-1}\right) < \varepsilon.$$

This means that (x_n) is slowly oscillating.
In order to apply theorem (4.9) we consider the following example.

Example 4.10:

Let C [0,1] be the set of all continuous functions defined on [0,1] and let $\|\cdot\|$ be the norm on C[0.1] given by $\|f\| = \max_{x\in[0,1]}|f(x)|$. Consider NNS $(C[0.1], \mu_0, \nu_0, \omega_0)$ where μ_0, ν_0 and ω_0 are as in NNS $(C[0.1], \mu_0, \nu_0, \omega_0)$ and $\{n(f_n - f_{n-1})\}$ is q- bounded in view of the facts that

$$\lim_{t\to\infty}\inf_{n\in\mathbb{N}} \mu_0(n(f_n - f_{n-1}), t) = \lim_{t\to\infty}\inf_{n\in\mathbb{N}} \frac{t}{t + \|n(f_n - f_{n-1})\|},$$

$$= \lim_{t\to\infty}\inf_{n\in\mathbb{N}} \frac{t}{t + \max_{x\in[0,1]}|n(f_n(x) - f_{n-1}(x))|},$$

$$= \lim_{t\to\infty}\inf_{n\in\mathbb{N}} \frac{t}{t + \max_{x\in[0,1]}|n(x^n - x^{n+1})|},$$

$$= \lim_{t\to\infty}\inf_{n\in\mathbb{N}} \frac{t}{t + \left(\frac{n}{n+1}\right)^{n+1}} = \lim_{t\to\infty} \frac{t}{t + 1/e} = 1,$$

$$\lim_{t\to\infty} \sup_{n\in\mathbb{N}} \nu_0(n(f_n - f_{n-1}), t) = \lim_{t\to\infty}\sup_{n\in\mathbb{N}} \frac{\|n(f_n - f_{n-1})\|}{t + \|n(f_n - f_{n-1})\|},$$

$$= \lim_{t\to\infty}\sup_{n\in\mathbb{N}} \frac{\max_{x\in[0,1]}|n(f_n(x) - f_{n-1}(x))|}{t + \max_{x\in[0,1]}|n(f_n(x) - f_{n-1}(x))|},$$

$$= \lim_{t\to\infty}\sup_{n\in\mathbb{N}} \frac{\max_{x\in[0,1]}|n(x^n - x^{n+1})|}{t + \max_{x\in[0,1]}|n(x^n - x^{n+1})|},$$

$$= \lim_{t\to\infty}\sup_{n\in\mathbb{N}} \frac{\left(\frac{n}{n+1}\right)^{n+1}}{t + \left(\frac{n}{n+1}\right)^{n+1}} = \lim_{t\to\infty} \frac{\frac{1}{e}}{t + \frac{1}{e}} = 0 \text{ and}$$

$$\lim_{t\to\infty} \sup_{n\in\mathbb{N}} \omega_0(n(f_n - f_{n-1}), t) = \lim_{t\to\infty}\sup_{n\in\mathbb{N}} \frac{\|n(f_n - f_{n-1})\|}{t},$$

$$= \lim_{t\to\infty}\sup_{n\in\mathbb{N}} \frac{\max_{x\in[0,1]}|n(f_n(x) - f_{n-1}(x))|}{t},$$

$$= \lim_{t\to\infty}\sup_{n\in\mathbb{N}} \frac{\max_{x\in[0,1]}|n(x^n - x^{n+1})|}{t},$$

$$= \lim_{t\to\infty}\sup_{n\in\mathbb{N}} \frac{\left(\frac{n}{n+1}\right)^{n+1}}{t} = \lim_{t\to\infty} \frac{1/e}{t} = 0 \text{ and}$$

Hence , sequence $\{f_n\}$ is slowly oscillating two sided Tauberian theorem due to in view of theorem (4.8) and (4.9).

5 . Conclusions:

In the current paper, as an introduction to summability theory and Tauberian theory in NNS, we have defined Cesaro Summability method in NNS, and proved a Tauberian theorem for Cesaro Summability method. Furthermore, we have introduced the concept of slowly oscillating sequence in NNS, given its relationship with q-bounded sequences and showed that slow oscillation and q-boundedness serves Tauberian conditions for Cesaro summability method in NNS. In view of the results of this paper, different types of summability methods can be defined in NNS to tackle problems where Cesaro method fails and the concepts of slow oscillation and q-boundedness in NNS can be used to obtain Tauberian results for other convergence methods.

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