JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1239-1247
https://publishoa.com
ISSN: 1309-3452

F-average eccentric graphs

T.Sathiyanandham[1] and S.Arockiaraj[2]

[1,2] Assistant Professor, Department of Mathematics,
Government Arts and Science College, Sivakasi-626 124, Tamilnadu, India.
Email id: sathiyan.maths@gmail.com, psarockiaraj@gmail.com.

ABSTRACT
The F-average eccentric graph $AE_F(G)$ of a graph G has the vertex set as in G and any two vertices u and v are adjacent in $AE_F(G)$ if either they are at a distance $\left\lfloor\frac{e(u)+e(v)}{2}\right\rfloor$ while G is connected or they are in different components while G is disconnected. A graph G is called a F-average eccentric graph if $AE_F(H) \cong G$ for some graph H. The main aim of this paper is to find a necessary and sufficient condition for a graph to be a F-average eccentric graph.

AMS (2000) subject classifications: 05C12

Keywords: F-average eccentric vertex, F-average eccentric graph labeling

1 Introduction

Throughout this paper, a graph means a non trivial simple graph. For other graph theoretic notation and terminology, we follow [7,8]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. $d(v)$ denotes the degree of a vertex $v \in V(G)$, the order of G is $|V(G)|$ and the size is $|E(G)|$. The distance $d(u,v)$ between a pair of vertices u and v is the length of a shortest path joining them. The eccentricity $e(u)$ of a vertex u is the distance to a vertex farthest from u. The radius $r(G)$ of G is the minimum eccentricity among the eccentricities of the vertices of G and the diameter $d(G)$ of G is the maximum eccentricity among the eccentricities of the vertices of G. Nestled in between is the average eccentricity; this was introduced by [7] (as eccentric mean). A graph G for which $r(G) = d(G)$ is called a self-centered graph of radius $r(G)$. Double star $S(n,m)$ was introduced by Grossman et al., [6]. The double star $S(n,m)$, where $n \geq m \geq 0$, is the graph consisting of the union of two stars $K_{1,n}$ and $K_{1,m}$ together with a line joinig their centers. A vertex v is called an eccentric vertex of a vertex u if $d(u,v) = e(u)$. A vertex v of G is called an eccentric vertex of G if it is the eccentric vertex of some vertex of G. Let $S_i(G)$ denote a subset of the vertex set of G such that $e(u) = i$ for all $u \in V(G)$. The eccentric graph [2] based on G is denoted by G_e whose vertex set is $V(G)$ and two vertices u and v are adjacent in G_e if $d(u,v) = min\{e(u), e(v)\}$. In [5], the radial graph $R(G)$ based on G has the vertex set as in G and two vertices are adjacent if the distance between them is equal to the radius of G when G is connected. If G is disconnected, then two vertices are adjacent in $R(G)$ if they are in different components of G. A graph G is called a radial graph if $R(H) = G$ for some graph H. In this paper, we introduce a new graph called F-average eccentric graph. Two vertices u and v of a graph are said to be F-average eccentric to each other if $d(u,v) = \left\lfloor\frac{e(u)+e(v)}{2}\right\rfloor$. The F-average eccentric graph of a graph G, denoted by $AE_F(G)$, has the vertex set as in G and any two vertices u and v are adjacent in $AE_F(G)$ if either they are at a distance $d(u,v) = \left\lfloor\frac{e(u)+e(v)}{2}\right\rfloor$ while G is connected or they are in different components while G is disconnected. A graph G is called a F-average eccentric graph if $AE_F(H) \cong G$ for some graph H. The notion of F-average eccentric graph is different from antipodal graph, eccentric graph and radial graph, since $S(2,1)$ is a F-average eccentric graph but not an antipodal graph, $P_4 \cup K_1$ is a F-average eccentric graph but not an eccentric graph and P_4 is a

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1239-1247
https://publishoa.com
ISSN: 1309-3452

F-average eccentric graph but not a radial graph. In this paper, we obtain a necessary and sufficient condition for a graph to be a F-average eccentric graph.

Theorem A[8] If $d(G) \geq 3$, then $d(\overline{G}) \leq 3$.

Theorem B[8] If $d(G) \geq 4$, then $d(\overline{G}) \leq 2$.

Theorem C[8] If $r(G) \geq 3$, then $r(\overline{G}) \leq 2$.

Theorem D[7] If $r(G) = d(G) \geq 3$, then $r(\overline{G}) = d(\overline{G}) = 2$.

Theorem E[5] For cycle C_n, $n \geq 4$, $R(C_n) = \frac{n}{2}K_2$ if n is even and $R(C_n) \cong C_n$ if n is odd.

Theorem F[5] Let G be a graph of order n. Then $R(G) = \overline{G}$ if and only if either $S_2(G) = V(G)$ or G is the union of complete graphs.

Let $F_{11}, F_{12}, F_{22}, F_{23}, F_{24}, F_3$ be the set of all connected graphs G for which $r(G) = d(G) = 1$, $r(G) = 1$ and $d(G) = 2$, $r(G) = d(G) = 2$, $r(G) = 2$ and $d(G) = 3$, $r(G) = 2$ and $d(G) = 4$, $r(G) \geq 3$ respectively and F_4 be the set of all disconnected graphs.

***F*-average eccentric graph of some classes of graphs**

Observation 2.1. If G is either a self centered graph or a disconnected graph, then $AE_F(G) = R(G) = A(G) = G_e$.

Proposition 2.2. Let P_n be any path on $n \geq 1$ vertices. Then

$$AE_F(P_n) = \begin{cases} P_n, & \text{if } n = 1,2 \\ C_n, & \text{if } n = 3 \\ P_4 \cup \overline{K}_{n-4}, & \text{if } n \geq 4. \end{cases}$$

Proof. When $n = 1,2, AE_F(P_n) = P_n$ and $AE_F(P_n) = C_n$ if $n = 3$. Let G be a path $v_1v_2v_3 \ldots v_n$ with $n \geq 4$ vertices. Then $e(v_i) = n - i$ for $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$, $e(v_i) = i - 1$ for $\left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n$ and $d(v_i, v_j) = j - i$ for $1 \leq i, j \leq n$. This implies that $d(v_1, v_n) = n - 1 = \left\lfloor \frac{e(v_1)+e(v_n)}{2} \right\rfloor$, $d(v_1, v_{n-1}) = n - 2 = \left\lfloor \frac{e(v_1)+e(v_{n-1})}{2} \right\rfloor$. Assume that $i < j$. If $1 \leq i, j \leq \left\lceil \frac{n}{2} \right\rceil$, then $d(v_i, v_j) = j - i = (n - i) - (n - j) < \left\lfloor \frac{e(v_i)+e(v_j)}{2} \right\rfloor$. If $2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$ and $\left\lceil \frac{n}{2} \right\rceil + 1 \leq j \leq n - 1$, then $\left\lfloor \frac{e(v_i)+e(v_j)}{2} \right\rfloor = \left\lfloor \frac{(n-i)+(j-1)}{2} \right\rfloor > j - i = d(v_i, v_j)$. If $j \neq n - 1$, then $d(v_1, v_j) = j - 1 < \left\lfloor \frac{(n-1)+(j-1)}{2} \right\rfloor = \left\lfloor \frac{e(v_1)+e(v_j)}{2} \right\rfloor$. By graph symmetry, the F-average eccentric pairs in G are (v_{n-1}, v_1), (v_1, v_n), (v_n, v_2) and the remaining pairs are not F-average eccentric pairs in G. These F-average eccentric pairs of vertices form the graph $AE_F(G)$. In $AE_F(G)$, $v_2v_nv_1v_{n-1}$ is a path on 4 vertices and the remaining vertices form $\overline{K}_{n-4}$.□

When $n \geq 6$ and n is even, $AE_F(P_n) = P_4 \cup \overline{K}_{n-4}$, $A(P_n) = P_2 \cup \overline{K}_{n-2}$ and $(P_n)_e = S(\frac{n}{2} - 1, \frac{n}{2} - 1)$ and $R(P_n) = \frac{n}{2}K_2$. So $AE_F(G)$ need not be isomorphic to $A(G), G_e$ and $R(G)$.

Proposition 2.3. Let C_n be any cycle on $n \geq 3$ vertices. Then

$$AE_F(C_n) \cong \begin{cases} \frac{n}{2}K_2, & \text{if } n \text{ iseven} \\ C_n, & \text{if } n \text{ isodd.} \end{cases}$$

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1239-1247
https://publishoa.com
ISSN: 1309-3452

Proof. Let $v_1, v_2, \dots, v_n$ be the n vertices of the cycle C_n. If $n = 3$, then $r(C_3) = 1$ and $e(v_i) = 1$ for $i = 1,2,3$. Hence $AE_F(C_3) \cong C_3$. If $n \geq 4$, then the result follows from Observation 2.1 and Theorem E. □

Observation 2.4. Let G_i be a connected graph with r_i vertices for $i = 1,2,\dots,n$. If G is the union of $G_1, G_2, \dots, G_n$, then $AE_F(G) = K_{r_1,r_2,\dots,r_n}$.

Proposition 2.5. $AE_F(K_{r_1,r_2,\dots,r_n}) = K_{r_1} \cup K_{r_2} \cup \dots \cup K_{r_n}$ where $r_1, r_2, \dots, r_n \geq 2$.
Proof. Let $V_1, V_2, \dots, V_n$ be the n partitions of $V(K_{r_1,r_2,\dots,r_n})$ for which $|V_1| = r_1, |V_2| = r_2, \dots, |V_n| = r_n$. Then $e(u) = 2$ for all $u \in V(K_{r_1,r_2,\dots,r_n})$. Let $u \in V_i$ for any $i = 1,2,\dots,n$. Then every vertex in $V_i - \{u\}$ is a F-average eccentric vertex of u and the remaining vertices of $K_{r_1,r_2,\dots,r_n}$ are the non F-average eccentric vertices of u. Hence $AE_F(K_{r_1,r_2,\dots,r_n}) = K_{r_1} \cup K_{r_2} \cup \dots \cup K_{r_n}$. □

Theorem 2.6. For any graph G on n vertices, a vertex is a full degree vertex in $AE_F(G)$ if and only if either it is an isolated vertex or a full degree vertex or a non full degree vertex adjacent to the full degree vertices only in G.
Proof. If v is an isolated vertex in G, then v is the full degree vertex in $AE_F(G)$. If v is a full degree vertex in G, then for any $uv \in E(G), |e(u) - e(v)| \leq 1$. This implies that $\left\lfloor \frac{e(u)+e(v)}{2} \right\rfloor = 1 = d(u,v)$ whenever $uv \in E(G)$. Therefore $uv \in E(AE_F(G))$ whenever $uv \in E(G)$. Hence v is the full degree vertex in $AE_F(G)$. Let G be a connected graph having full degree vertices. If w is a non full degree vertex adjacent to any of full degree vertices in G, then $d(w,u) = 2 = \left\lfloor \frac{e(w)+e(u)}{2} \right\rfloor$ for each non full degree vertex u in G and $d(w,v) = 1 = \left\lfloor \frac{e(w)+e(v)}{2} \right\rfloor$ for each full degree vertex v in G. Then w is a full degree vertex in $AE_F(G)$

Suppose v is a full degree vertex in $AE_F(G)$. If G is a disconnected graph having m components say $H_1, H_2, \dots, H_m$ with $|H_i| = n_i > 1$ for $i = 1,2,\dots,m$, then by Observation 2.4, $AE_F(G) = K_{n_1,n_2,\dots,n_m}$, a contracdiction. Hence v is an isolated vertex in G. Let G be a connected graph with no full degree vertex. Then $e(u) \geq 2$ for all $u \in V(G)$. Therefore $uv \notin E(AE_F(G))$ whenever $uv \in E(G)$. Thus $AE_F(G)$ has no full degree vertex, a contradiction. Hence G should have a full degree vertex. Let w be a full degree vertex in G. Suppose v is not a full degree vertex in G. Then $vw \in E(G)$. If v is adjacent to at least one non full degree vertex u in G, then $vu \notin E(AE_F(G))$, a contradiction to the fact that v is a full degree vertex in $AE_F(G)$. Thus v is adjacent to the full degree vertices only in G. □

Theorem 2.7. Let G be a graph on n vertices. If G has $r \geq 1$ number of full degree vertices $v_1, v_2, \dots, v_r$, then $AE_F(G) = K_r + (\overline{G - \{v_1, v_2, \dots, v_r\}})$.
Proof. Let $v_1, v_2, \dots, v_r, v_{r+1}, \dots, v_n$ be the vertices of G and let $wv \in E(AE_F(G))$. If either w or $v \in \{v_1, v_2, \dots, v_r\}$, then $wv = wv_j \in E(G)$ for some j. If none of w and v is in $\{v_1, v_2, \dots, v_r\}$, then $e(w) = e(v) = 2$. Since $wv \in E(AE_F(G)), wv \notin E(G)$. Therefore $wv \notin E(G - \{v_1, v_2, \dots, v_r\})$ and hence $wv \in E(\overline{G - \{v_1, v_2, \dots, v_r\}})$

Suppose $wv \in E(K_r + (\overline{G - \{v_1, v_2, \dots, v_r\}})) = E(\overline{G - \{v_1, v_2, \dots, v_r\}}) \cup \{uv_j \in E(G) : u \in V(G), 1 \leq j \leq r\}$. If $wv = uv_i$ for $u \in V(G)$ and $v_i \in \{v_1, v_2, \dots, v_r\}$, then $d(u, v_i) = 1 = \left\lfloor \frac{e(u)+e(v_i)}{2} \right\rfloor$ and hence $wv \in E(AE_F(G))$. If $wv \in E(\overline{G - \{v_1, v_2, \dots, v_r\}})$, then $wv \notin E(G - \{v_1, v_2, \dots, v_r\})$. This implies that $wv \notin E(G)$. Then $d(w,v) = 2 = \left\lfloor \frac{e(w)+e(v)}{2} \right\rfloor$ and hence $wv \in E(AE_F(G))$. Thus $AE_F(G) = K_r + (\overline{G - \{v_1, v_2, \dots, v_r\}})$. □

Corollary 2.8. If $F_{m,n} = \overline{K}_m + P_n$ is a fan graph on $m + n \geq 2$ vertices for any positive integers m and n, then

$$AE_F(F_{m,n}) = \begin{cases} K_{n+1}, & \text{if } n = 1,2,3 \text{ and } m = 1 \\ K_1 + \overline{P}_n, & \text{if } n > 3 \text{ and } m = 1 \\ K_{m+n}, & \text{if } n = 1,2 \text{ and } m > 1 \\ (K_m \cup P_2) + K_1, & \text{if } n = 3 \text{ and } m > 1 \\ K_m \cup \overline{P}_n, & \text{if } n > 3 \text{ and } m > 1. \end{cases}$$

Proof. Follows from Theorem 2.7. □

Corollary 2.9. Let $W_n = K_1 + C_n$ be a wheel graph on $n \geq 3$ vertices. Then $AE_F(W_n) = K_1 + \overline{C}_n$.
Proof. If $r = 1$, then by Theorem 2.7, $AE_F(W_n) = K_1 + \overline{C}_n$. □

Theorem 2.10. Let G be a graph. Then $AE_F(G) = K_m + K_{r_1,r_2,\dots,r_l}$ for any positive integers m, l, r_i and $1 \leq i \leq l$ if and only if any one of the following holds
(1) G is disconnected with exactly $l + m$ components and it has at least m isolated vertices
(2) G is connected and it has m full degree vertices so that the deletion of these full degree vertices in G forms a disconnected graph with l components in which each component is complete.
Proof. If (1) holds, then by Observation 2.4, $AE_F(G) = K_m + K_{r_1,r_2,\dots,r_l}$ for any positive integers m, l, r_i and $1 \leq i \leq l$. Suppose (2) holds. Let $\{v_1, v_2, \dots, v_s\}$ be the set of all full degree vertices of G where $s \geq m$. Then $G - \{v_1, v_2, \dots, v_s\}$ has at most l complete components, say $K_{r_1}, K_{r_2}, \dots, K_{r_l}$. By Theorem 2.6, each v_i is a full degree vertex in $AE_F(G)$. Let $x, y \in V(K_{r_i})$ and $z \in V(K_{r_j})(j \neq i)$. Then $e(x) = e(y) = e(z) = 2$. Since $xy \in E(K_{r_i})$ and $xz \notin E(G)$, $xy \notin E(AE_F(G))$ and $xz \in E(AE_F(G))$. Hence $AE_F(G) = K_m + K_{r_1,r_2,\dots,r_l}$ for any positive integers m, l, r_i and $1 \leq i \leq l$.

Suppose $AE_F(G) = K_m + K_{r_1,r_2,\dots,r_l}$ for any positive integers m, l, r_i and $1 \leq i \leq l$. Assume that $r_1 \leq r_2 \leq \dots \leq r_l$. If G is disconnected having no isolated vertex, by Observation 2.4, $AE_F(G)$ is a complete $t(\geq 2)$-partite graph having no full degree vertex, a contradiction. If G is disconnected with at most $m - 1$ isolated vertices, then by Theorem 2.6, $AE_F(G)$ has at most $m - 1$ full degree vertices, a contradiction. So G should have at least m isolated vertices. Let $H_1, H_2, \dots, H_m, H_{m+1}, \dots, H_t$ be the components of G so that $|H_i| = 1$ for $1 \leq i \leq m$, $|H_{m+i}| = s_i$ for $1 \leq i \leq t - m$ and $s_1 \leq s_2 \leq \dots \leq s_{t-m}$. If $t > m + l$ $(< m + l)$, then by Observation 2.4, $AE_F(G) = K_m + K_{s_1,s_2,\dots,s_{m+l},s_{m+l+1},\dots,s_t}$ $(K_m + K_{s_1,s_2,\dots,s_{t-m}})$, a contradiction to the assumption of $AE_F(G)$. In G, if $t = m + l$ and $r_j \neq s_j$ for some j, then it arises a contradiction to the assumption of $AE_F(G)$. If G is connected with no full degree vertex, then by Theorem 2.6, $AE_F(G)$ has no full degree vertex, a contradiction. If the number of full degree vertices in G is fewer than m, then by Theorem 2.6, $AE_F(G)$ has at most $m - 1$ full degree vertices, a contradiction. Therefore G should have at least m full degree vertices. Let $\{v_1, v_2, \dots, v_s\}$ be the set of all full degree vertices of G where $s \geq m$. Take $G_0 = G - \{v_1, v_2, \dots, v_s\}$. If G_0 is complete, then all the vertices of G are full degree vertices, by Theorem 2.7, $AE_F(G)$ is complete, a contradiction. If G_0 is connected and non complete, by Theorem 2.7, $AE_F(G) = K_s + \overline{G_0}$, a contradiction. So G_0 is disconnected. If G_0 has a non complete component H, then by Theorem 2.7, $E(\overline{H}) \subseteq E(AE_F(G))$, a contradiction. Hence each component of G_0 is complete. Let $H_1, H_2, \dots, H_t(t > l)$ be the complete components of G_0 with $|H_i| = r_i$ for $1 \leq i \leq t$. Since eccentricity of each vertex in H_i is 2, $xy \in E(AE_F(G))$ for all $x \in V(H_i), y \in V(H_j), i \neq j$ and $1 \leq i, j \leq t$. Therefore $K_{r_1,r_2,\dots,r_l,r_{l+1},\dots,r_t}$ is a subgraph of $AE_F(G)$, a contradiction. Thus G_0 has at most l components in which each component is complete. □

Proposition 2.11. Let $L_n = P_n \times P_2$ be a ladder with $n \geq 1$ steps. Then

$$AE_F(L_n) \cong \begin{cases} K_2, & \text{if } n = 1 \\ 2K_2, & \text{if } n = 2 \\ C_6, & \text{if } n = 3 \\ 2P_4 \cup \overline{K}_{2(n-4)}, & \text{if } n \geq 4. \end{cases}$$

Proof. Let $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ be the vertices on the path in L_n of length $n - 1$. If $n = 1$, then $L_1 = K_2$ and $AE_F(L_1) \cong K_2$. If $n = 2$, then $L_2 = C_4$ and $AE_F(L_2) \cong 2K_2$. If $n = 3$, then the F-average eccentric pairs in L_3 are $(u_1, v_3), (v_3, u_2), (u_2, v_1)$, $(v_1, u_3), (u_3, v_2)$ and (v_2, u_1). Hence $AE_F(L_3) \cong C_6$. Let G be a ladder with $n \geq 4$ steps. Then $e(u_i) = e(v_i) = n + 1 - i$ for $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$, $e(u_i) = e(v_i) = i$ for $\left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n$, $d(u_i, u_j) = d(v_i, v_j) = j - i$ and $d(u_i, v_j) = |j - i| + 1$. This implies that $d(u_1, v_n) = n =$

$\left\lfloor\frac{e(u_1)+e(v_n)}{2}\right\rfloor$ and $d(u_1,v_{n-1}) = n-1 = \left\lfloor\frac{e(u_1)+e(v_{n-1})}{2}\right\rfloor$. Assume that $i<j$. If $1 \le i,j \le \left\lceil\frac{n}{2}\right\rceil$, then $d(u_i,u_j) = j-i < \left\lfloor\frac{(n+1-i)+(n+1-j)}{2}\right\rfloor = \left\lfloor\frac{e(u_i)+e(u_j)}{2}\right\rfloor$ and $d(u_i,v_j) = |j-i|+1 < \left\lfloor\frac{(n+1-i)+(n+1-j)}{2}\right\rfloor = \left\lfloor\frac{e(u_i)+e(v_j)}{2}\right\rfloor$. If $2 \le i \le \left\lceil\frac{n}{2}\right\rceil$ and $\left\lceil\frac{n}{2}\right\rceil + 1 \le j \le n-1$, then $d(u_i,u_j) = j-i < \left\lfloor\frac{(n+1-i)+j}{2}\right\rfloor = \left\lfloor\frac{e(u_i)+e(u_j)}{2}\right\rfloor$ and $d(u_i,v_j) = |j-i|+1 < \left\lfloor\frac{(n+1-i)+j}{2}\right\rfloor = \left\lfloor\frac{e(u_i)+e(v_j)}{2}\right\rfloor$. If $j \ne n-1$, then $d(u_1,u_j) = j-1 < \left\lfloor\frac{n+j}{2}\right\rfloor = \left\lfloor\frac{e(u_1)+e(u_j)}{2}\right\rfloor$, $d(u_1,v_j) = j < \left\lfloor\frac{n+j}{2}\right\rfloor = \left\lfloor\frac{e(u_1)+e(v_j)}{2}\right\rfloor$. If $i=j$, then $d(u_i,v_j) < \left\lfloor\frac{e(u_i)+e(v_j)}{2}\right\rfloor$. By graph symmetry, the F-average eccentric pairs in G are $(v_{n-1},u_1),(u_1,v_n),(v_n,u_2),(u_{n-1},v_1),(v_1,u_n),(u_n,v_2)$ and the remaining pairs in G are not F-average eccentric pairs in G. Let $P_1{:}\, v_{n-1}u_1v_nu_2$ and $P_2{:}\, u_{n-1}v_1u_nv_2$ be two paths. Thus $AE_F(G)$ is the union of P_1, P_2 and $\overline{K}_{2(n-4)}$. □

3 *F*-average eccentric graphs

Proposition 3.1. If $r(G) \ge 2$, then $AE_F(G) \subseteq \overline{G}$.
Proof. By the definition, $V(AE_F(G)) = V(\overline{G}) = V(G)$. If $e = uv \in E(AE_F(G))$ but does not belong to $E(\overline{G})$, then $uv \in E(G)$ and by the definition, either $e(u) = 1$ or $e(v) = 1$, a contradiction. Therefore $E(AE_F(G)) \subseteq E(\overline{G})$ and hence $AE_F(G) \subseteq \overline{G}$. □

Theorem 3.2. For any graph G on n vertices, $AE_F(G) = G$ if and only if $G \in F_{11}$.
Proof. Suppose $AE_F(G) = G$. If G is disconnected with $r \ge 2$ components, then by Observation 2.4, $AE_F(G)$ is a complete r partite graph, a contradiction. So G is connected. If $r(G) \ge 2$, then by Proposition 3.1, $AE_F(G) \subseteq \overline{G}$, a contradiction. If $G \in F_{12}$, then by Theorem 2.7, $AE_F(G) \ne G$, a contradiction.
Suppose $G \in F_{11}$. Then by Theorem 2.10, $AE_F(G) = K_n = G$. □

Proposition 3.3. For any graph $G \in F_{22}, AE_F(G) = \overline{G}$.
Proof. Follows from Observation 2.1 and Theorem F. □

Proposition 3.4. Every complete graph G of order $n \ge 1$, is a F-average eccentric graph.
Proof. Follows from Theorem 2.10. □

Proposition 3.5. $K_l + \overline{K}_n$ is a F-average eccentric graph, for any positive integers l and n.
Proof. Follows from Theorem 2.10. □

Proposition 3.6. $AE_F(G) = K_m + K_{r_1,r_2,\ldots,r_l}$ is a F-average eccentric graph, for any positive integers m, l, r_i and $1 \le i \le l$.
Proof. Follows from Theorem 2.10. □

Proposition 3.7. Every path P_n is a F-average eccentric graph, for any positive integer n.
Proof. When $n = 1,2, P_n$ is a F-average eccentric graph of itself. When $n = 3, AE_F(K_1 \cup K_2) = P_3$. Let $v_1v_2v_3v_4$ be a P_4 path. Then the eccentricities of v_1,v_2,v_3,v_4 in P_4 are 3,2,2,3 respectively and the eccentricities of v_1,v_2,v_3,v_4 in $\overline{P}_4$ are 2,3,3,2 respectively. The non adjacent pairs in $\overline{P}_4$ are $(v_1,v_2),(v_2,v_3),(v_3,v_4)$ and $\overline{P}_4 = v_2v_4v_1v_3$. The F-average eccentric pairs in $\overline{P}_4$ are $(v_2,v_3),(v_1,v_2),(v_3,v_4)$. Hence $AE_F(\overline{P}_4) = P_4$. Assume that $n \ge 5$. Let $v_1v_2\ldots v_n$ be a path P_n. For each $i, 2 \le i \le n-1, v_i$ is non adjacent to v_{i-1} and v_{i+1} in $\overline{P}_n$ and it is adjacent to all other vertices in $\overline{P}_n$. Since $d(P_n) = n-1$, by Theorem B, $d(\overline{P}_n) \le 2$. Also $\overline{P}_n$ has no full degree vertex. So $\overline{P}_n \in F_{22}$ and by Proposition 3.3, $AE_F(\overline{P}_n) = P_n$. □

Proposition 3.8. Every cycle C_n is a F-average eccentric graph, for any positive integer $n \geq 3$.
Proof. If $n = 3$ and $H = \overline{K}_3$ or P_3 or C_3, then by Theorem 2.10, Proposition 2.2 and Proposition 2.3, $AE_F(H) = C_3$. If $n = 4$, then $AE_F(P_2 \cup P_2) = AE_F(\overline{C}_4) = C_4$. Assume that $n > 4$. For a cycle $C_n, e(u) = 2$ for all $u \in V(\overline{C}_n)$. By Proposition 3.3, $AE_F(\overline{C}_n) = C_n$. Hence C_n is a F-average eccentric graph. □

Proposition 3.9. Every ladder L_n with n steps is a F-average eccentric graph, for any positive integer n.
Proof. When $n = 1, AE_F(L_n) = L_n$. Asseume that L_n is a ladder with $n \geq 2$ steps. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices of the paths of length $n-1$ in L_n. Then the non adjacent pairs in $\overline{L}_n$ are $(u_1, v_1), (u_n, v_n), (u_{i-1}, u_i), (u_i, u_{i+1}), (u_i, v_i)$, $(v_{i-1}, v_i), (v_i, v_{i+1})$ for all $i = 2,3, \ldots, n-1$. Then $\overline{L}_n \in F_{22}$ and by Proposition 3.3, $AE_F(\overline{L}_n) = L_n$. Hence L_n is a F-average eccentric graph. □

Proposition 3.10. Every wheel graph W_n on $n+1$ vertices is a F-average eccentric graph, for any positive integer $n \geq 3$.
Proof. Consider the graph $H = K_1 + \overline{C}_n$. Let u be the vertex of K_1 and let $v_1, v_2, \ldots, v_n$ be the vertices of $\overline{C}_n$. Then $e(u) = 1$ and $e(v_i) = 2$ in H for all $i = 1,2, \ldots, n$. Since $d(u, v_i) = 1 = \left\lfloor \frac{e(u)+e(v_i)}{2} \right\rfloor, uv_i \in AE_F(H)$, for $i = 1,2, \ldots, n$. Since $e(v_i) = 2$ for each $i, v_i v_j \in E(AE_F(H))$ if and only if $v_i v_j \notin E(H)$ for all $i \neq j$ and $1 \leq i, j \leq n$. Thus $AE_F(H) = K_1 + C_n = W_n$. □

Proposition 3.11. Every fan graph $F_{m,n} = \overline{K}_m + P_n$ is a F-average eccentric graph, for any positive integers m and $n = 1,2$.
Proof. Follows from Theorem 2.10. □

Proposition 3.12. Every complete n-partite graph is a F-average eccentric graph, for any positive integer $n \geq 2$.
Proof. Follows from Observation 2.4. □

4 A necessary and sufficient condition for a graph to be a F-average eccentric graph

Theorem 4.1. Let G be a graph and let $S_3(G)$ be the set of all vertices of $V(G)$ whose eccentricities are 3. Then $AE_F(G) = \overline{G}$ if and only if any one of the following conditions hold
(1) $G \in F_{22}$
(2) $G \in F_{23}$ and there is no vertex adjacent to atleast two non adjacent vertices in $S_3(G)$
(3) G is disconnected having no non-complete component.
Proof. If (1) holds, then by Proposition 3.3, $AE_F(G) = \overline{G}$. Suppose (2) holds. Since $r(G) = 2$, by Proposition 3.1, $AE_F(G) \subseteq \overline{G}$. Let $uv \in E(\overline{G})$. where $u, v \in S_3(G)$. By hypothesis, $d_G(u, v) = 3 = \left\lfloor \frac{e(u)+e(v)}{2} \right\rfloor$ and hence $uv \in E(AE_F(G))$. Therefore $AE_F(G) = \overline{G}$. Suppose (3) holds. If G is totally disconnected, then by the definition, $AE_F(G) = K_n = \overline{G}$. Suppose G has at least one component H with $|H| \geq 2$. Then by Observation 2.1 and Theorem F, $AE_F(G) = \overline{G}$.

Suppose $AE_F(G) = \overline{G}$. If $G \in F_{11} \cup F_{12}$. Then by Theorem 2.7 and Theorem 3.2, $AE_F(G) \neq \overline{G}$. Suppose $G \in F_{23}$ and there is a vertex $u \in V(G)$ in which it is adjacent to the non adjacent pair of vertices v and $w \in S_3(G)$. Since $d(v, w) = 2$ and $\left\lfloor \frac{e(v)+e(w)}{2} \right\rfloor = 3, vw \notin E(AE_F(G))$, a contradiction. If $G \in F_{24}$, then there exists two vertices u and v such that $e(u) = 2$ and $e(v) = 4$. But $d(u, v) = 2$ and $\left\lfloor \frac{e(u)+e(v)}{2} \right\rfloor = 3$. Therefore $uv \in E(\overline{G})$ but $uv \notin E(AE_F(G))$, a contradiction. If $G \in F_3$, $e(u) \geq 3$ for all $u \in V(G)$. Then there is a pair of vertices v and w such that $d(v, w) = 2$ but $\left\lfloor \frac{e(v)+e(w)}{2} \right\rfloor \geq 3$. Therefore $vw \in E(\overline{G})$ but $vw \notin E(AE_F(G))$, a contradiction. If G is disconnected with at least one non complete component H, then every pair of non adjacent vertices u and v in H are adjacent in $\overline{G}$. But by the definition, $uv \notin E(AE_F(G))$, a contradiction.

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1239-1247
https://publishoa.com
ISSN: 1309-3452

□

Corollary 4.2. If G and $\overline{G} \in F_{22}$, then G and $\overline{G}$ are F-average eccentric graphs.
Proof. By theorem 4.1, $AE_F(G) = \overline{G}$. and $AE_F(\overline{G}) = G$. □

Corollary 4.3. Let G be any graph such that $\overline{G} \in F_{23}$. If there is no vertex adjacent to at least two non adjacent vertices in $S_3(\overline{G})$, then $AE_F(\overline{G}) = G$.
Proof. By theorem 4.1, the result follows. □

Corollary 4.4. If $r(G) > 1$ and $\overline{G}$ is disconnected with each component complete, then G is a F average eccentric graph.
Proof. By theorem 4.1, $AE_F(\overline{G}) = G$. □

Lemma 4.5. If G is disconnected, then $\overline{AE_F(G)}$ is also a disconnected graph with each component complete.
Proof. By Observation 2.4, the result follows. □

Theorem 4.6. If $r(G) \geq 2$ and $\overline{G}$ is disconnected with at least one non complete component, then G is not a F-average eccentric graph.
Proof. Suppose there exists a graph H such that $AE_F(H) = G$. If H is disconnected, then by Lemma 4.5, $\overline{AE_F(H)}$ is disconnected in which each component is complete, a contradiction to the fact that $\overline{G}$ is connected with at least one non complete component. Hence H must be connected. If $r(H) = 1$ and $d(H) = 1$, then $AE_F(H) = H = G$, a contradiction to $r(G) > 1$. If $r(H) = 1$ and $d(H) = 2$, then by the definition of F-average eccentric graph, $r(AE_F(H)) = 1$, a contradiction to $r(G) > 1$. So $r(H) > 1$. By Proposition 3.1, $AE_F(H) \subseteq \overline{H}$. Hence H is isomorphic to a spanning subgraph of $\overline{G}$. Since $\overline{G}$ is disconnected, H is disconnected, a contradiction to $r(G) \geq 2$. From these, we conclude that $AE_F(H)$ is not equal to G, a contradiction. □

Theorem 4.7. If $G \in F_{22}$ and $\overline{G} \in F_{23}$ in which there is a vertex w such that w is adjacent to at least two non adjacent pairs of vertices u and v in $S_3(\overline{G})$, then G is not a F-average eccentric graph.
Proof. Suppose there exists a graph H such that $AE_F(H) = G$. If H is disconnected, then by Lemma 4.5, $\overline{AE_F(H)}$ is disconnected, a contradiction to $\overline{G}$ is connected. Hence H must be connected. If $r(H) = 1$ and $d(H) = 1$, then $AE_F(H) = H = G$, a contradiction to $r(G) > 1$. If $r(H) = 1$ and $d(H) = 2$, then by the definition of F-average eccentric graph, $r(AE_F(H)) = 1$, a contradiction to $r(G) > 1$. So $r(H) > 1$. By Proposition 3.1, $AE_F(H) \subseteq \overline{H}$. Hence H is isomorphic to a spanning subgraph of $\overline{G}$. Since $r(\overline{G}) = 2$ and $d(\overline{G}) = 3, r(H) \geq 2$ and $d(H) \geq 3$. If w is a vertex adjacent to two non adjacent pairs u and v having eccentricities 3 in $\overline{G}$, then $e(w) \geq 2, e(u), e(v) \geq 3$ in H and $d_{\overline{G}}(u,v) = 2$. Since H is connected, $d(u,v) = 2 < \left\lfloor \frac{e(u)+e(v)}{2} \right\rfloor$ in H and hence u and v are non adjacent in $AE_F(H)$. From these, we conclude that $AE_F(H)$ is not equal to G, a contradiction. □

Theorem 4.8. If $G \in F_{22}$ and $\overline{G} \in F_{24}$, then G is not a F-average eccentric graph.
Proof. Suppose there exists a graph H such that $AE_F(H) = G$. If H is disconnected, then by Lemma 4.5, $\overline{AE_F(H)}$ is disconnected, a contradiction to $\overline{G}$ is connected. Hence H must be connected. If $r(H) = 1$ and $d(H) = 1$, then $AE_F(H) = H = G$, a contradiction to $r(G) > 1$. If $r(H) = 1$ and $d(H) = 2$, then by the definition of F-average eccentric graph, $r(AE_F(H)) = 1$, a contradiction to $r(G) > 1$. So $r(H) > 1$. By Proposition 3.1, $AE_F(H) \subseteq \overline{H}$. Hence H is isomorphic to a spanning subgraph of $\overline{G}$. Since $r(\overline{G}) = 2$ and $d(\overline{G}) = 4, r(H) \geq 2$ and $d(H) \geq 4$. Let $u \in V(H)$. Then u is adjacent to all the vertices v in G such that $d_{\overline{G}}(u,v) \geq 2$. $d_H(u,v) \geq 2$ whenever $d_{\overline{G}}(u,v) \geq 2$. But u is not adjacent to a vertex v in $AE_F(H)$ if

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1239-1247
https://publishoa.com
ISSN: 1309-3452

$d(u,v) < \left\lfloor \frac{e(u)+e(v)}{2} \right\rfloor$ in H. That is, for some pair of non adjacent vertices u and v in H, $uv \not\in E(AE_F(H))$, a contradiction to $uv \in E(G)$. From these, we conclude that $AE_F(H)$ is not equal to G, a contradiction. □

Using the same proof technique used in Theorem 4.8, we prove the following propositions

Proposition 4.9. If $G \in F_{22}$ and $\overline{G} \in F_3$, then G is not a F-average eccentric graph.

Proposition 4.10. If $G \in F_{23}$ and $\overline{G} \in F_{23}$ in which there is a vertex w such that w is adjacent to atleast two non adjacent pairs of vertices u and v in $S_3(\overline{G})$, then G is not a F-average eccentric graph.

Corollary 4.11. If $G \in F_{23}$ and $\overline{G} \in F_{23}$ in which there is no vertex adjacent to atleast two non adjacent vertices in $S_3(\overline{G})$, then G is a F-average eccentric graph.
Proof. By Theorem 4.1, $AE_F(\overline{G}) = G$. □

Corollary 4.12. If $G \in F_{24}$, then G is a F-average eccentric graph.
Proof. By Theorem B, $\overline{G} \in F_{22}$. Then by Theorem 4.1, $AE_F(\overline{G}) = G$. □

Corollary 4.13. If $G \in F_3$, then G is a F-average eccentric graph.
Proof. Since $G \in F_3, r(G) \geq 3$ and $d(G) \geq 3$. If $r(G) = d(G) = 3$, then by Theorem C, $\overline{G} \in F_{22}$. If $r(G) = 3$ and $d(G) > 3$ or $r(G) > 3$ and $d(G) > 3$, then by Theorems B and C, $\overline{G} \in F_{22}$. Then by Theorem 4.1, $AE_F(\overline{G}) = G$. □

Theorem 4.14. If G is a totally disconnected graph on $n \geq 2$ vertices, then G is not a F-average eccentric graph.
Proof. Suppose there exists a graph H such that $AE_F(H) = G$. If H is a disconnected graph, then by Lemma 4.5, $\overline{AE_F(H)}$ is disconnected, a contradiction to $\overline{G} \in F_{11}$. So H must be connected. If $H \in F_{11}$, then by Theorem 3.2, $AE_F(H) = H = G$, a contradiction to $G \in F_4$. If $H \in F_{12}$, then by Theorem 2.6, $AE_F(H) \in F_{11} \cup F_{12}$, a contradiction to $G \in F_4$. So $r(H) \geq 2$. Then every pair of F-average eccentric vertices in G are adjacent in $AE_F(H)$. Hence $AE_F(H)$ has an edge, a contradiction. Thus G is not a F-average eccentric graph. □

Theorem 4.15. If either G or $\overline{G} \not\in F_{12}$, then G is a F-average eccentric graph if and only if G is a F-average eccentric graph of itself or its complement.
Proof. Suppose either G is a F-average eccentric graph of itself or its complement. Then by the definition, G is a F-average eccentric graph.

Suppose G is a F-average eccentric graph.

Case 1. G is connected and $\overline{G}$ is connected.
Subcase 1.1. Suppose $G \in F_{22}$. Then by Theorem 4.8 and Proposition 4.9, $\overline{G} \not\in F_{24}$ and $\overline{G} \not\in F_3$. Therefore $\overline{G} \in F_{22}$ or $\overline{G} \in F_{23}$. If $\overline{G} \in F_{22}$, then by Theorem 4.1, $AE_F(\overline{G}) = G$. If $\overline{G} \in F_{23}$, then by Theorem 4.7, $\overline{G}$ has no vertex adjacent to at least two non adjacent vertices in $S_3(\overline{G})$, then by Corollary 4.3, $AE_F(\overline{G}) = G$.
Subcase 1.2. Suppose $G \in F_{23}$. Then $\overline{G} \in F_{22}$ or $\overline{G} \in F_{23}$. If $\overline{G} \in F_{22}$, then by Theorem 4.1, $AE_F(\overline{G}) = G$. If $\overline{G} \in F_{23}$, then by Proposition 4.10, $\overline{G}$ has no vertex adjacent to at least two non adjacent vertices in $S_3(\overline{G})$, then by Corollary 4.3, $AE_F(\overline{G}) = G$.
Subcase 1.3. Suppose $G \in F_{24}$. Then $\overline{G} \in F_{22}$ and by Theorem 4.1, $AE_F(\overline{G}) = G$.
Subcase 1.4. Suppose $G \in F_3$. Then $\overline{G} \in F_{22}$ and by Theorem 4.1, $AE_F(\overline{G}) = G$.
Case 2. G is connected and $\overline{G}$ is disconnected.
Subcase 2.1. Suppose $G \in F_{11}$. Then by Theorem 3.2, $AE_F(G) = G$.
Subcase 2.2. Suppose $r(G) \geq 2$. Then by Theorem 4.6, each component of $\overline{G}$ is complete. By Theorem 4.1, $AE_F(\overline{G}) = G$.

Case 3. G is disconnected. Then by assumption, either $\overline{G} \in F_{11}$ or $\overline{G} \in F_{22}$. If $\overline{G} \in F_{11}$, then G is totally disconnected and by Theorem 4.14, G is not a F-average eccentric graph, a contradiction. Hence $\overline{G} \in F_{22}$. By Theorem 4.1, $AE_F(\overline{G}) = G$. □

Theorem 4.16. Let $G \in F_{12}$ be a graph on n vertices. Then G is not a F-average eccentric graph if and only if it has a triangle and a pendant vertex.

Proof. Suppose G is not a F-average eccentric graph. Assume that G has no pendant vertex. Let $u_1, u_2, \ldots, u_r$ be the full degree vertices and $u_{r+1}, u_{r+2}, \ldots, u_n$ be the non full degree vertices. Then $e(v_i) = 1$ for $1 \leq i \leq r$ and $e(u_j) = 2$ for $r+1 \leq j \leq n$. Construct a graph H from G as follows. H has a vertex set as in G. Let $u_1, u_2, \ldots, u_r$ be the full degree vertices in H. Every pair of the non full degree vertices $u_{r+1}, u_{r+2}, \ldots, u_n$ are adjacent in H whenever they are non adjacent in G and vice versa. Then by the definition, $AE_F(H) = G$, a contradiction. Suppose G has no triangle. Then G has only one full degree vertex and the remaining vertices are pendant vertices. Then the graph H is $K_1 \cup K_{n-1}$ so that $AE_F(H) \cong G$, a contradiction.

Suppose G has a triangle and a pendant vertex. Suppose there exists a graph H such that $AE_F(H) = G$. Let x be a central vertex of G, u be a pendant vertex and xyz be a triangle in G. Then $e(u) = 2, e(x) = 1$ and $e(y) = 2 = e(z)$. If $w \neq x$ is another full degree vertex in G, then w is adjacent to u in G. a contradiction. Therefore x is the only full degree vertex in $AE_F(H)$. By Theorem 2.6, either x is an isolated vertex or a full degree vertex or a non full degree vertex adjacent to the full degree vertices in H. If x is an isolated vertex in H, then u is adjacent to all the vertices other than x in H. By the definition, $G = K_{1,n-1}$, a contradiction. If x is a full degree vertex in H, then u is adjacent to x as well as its eccentric vertices in $AE_F(H)$, a contradiction. If x is a non full degree vertex adjacent to the full degree vertex v in H, then u is adjacent to both x and v in $AE_F(H)$, a contradiction. □

References

[1] R.R. Singleton, *There is no irregular moore graph*, Amer. Math. Monthly, 75 (1968), 42 - 43.
[2] J. Akiyama, K. Ando, D. Avis, *Eccentric graphs*, Discrete math., 16 (1976), 187 - 195.
[3] R. Aravamuthan and B. Rajendran, *Graph equations involving antipodal graphs*, Presented at the seminar on combinatories and applications held at ISI, Culcutta during 14 - 17 December, (1982), 40 - 43.
[4] R. Aravamuthan and B. Rajendran, *On antipodal graphs*, Discrete math., 49 (1984), 193 –195.
[5] KM. Kathiresan and G. Marimuthu, *A study on radial graphs*, Ars Combin., 96 (2010), 353 – 360.
[6] J.W. Grossman, F. Harary and M. Klawe, *Generalized ramsey theory for graphs, x:double stars*, Discrete math., 28 (1979), 247 - 254.
[7] F. Buckley and F. Harary, *Distance in graphs*, Addition-wesley, Reading (1990).
[8] D.B. West, *Introduction to graph theory*, Prentice - Hall of india, New Delhi (2003).