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A Family of New Distance Models for Discrete Fuzzy Distributions and their detailed Properties

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ABSTRACT

The distance measures are incredibly imperative and play a fundamental responsibility towards optimization problems in discrete probability spaces. But, where probabilistic measures do not work, one can travel around the possibility of divergence measures in fuzzy distributions. This communiqué has been advocated for the generation of a family of original divergence models for discrete fuzzy distributions corresponding to the existing probabilistic measures and studied their comprehensive properties for proving their legitimacy.

Keywords: Fuzzy sets, Crisp sets, Probabilistic divergence, Fuzzy divergence, Convex function.

INTRODUCTION

In the literature on fuzzy spaces, it is the well-recognized actuality that uncertainty and fuzziness are the indispensable temperaments of individual thoughts and abundant existent world objectives. Fuzziness originated from our assessments and our language, and through the advancement, we practice information. The foremost exploitation of information is to eliminate uncertainty and fuzziness. In reality, we measure information abounding by the quantity of probabilistic uncertainty eradicated in experimentation and the measure of uncertainty eradicated. This also entitles an information model, although fuzziness models measure the indistinctness and haziness of uncertainties.

The commencement of fuzzy sets anticipated by Zadeh [21] from the point of view to grasp indecisive problems prepared the theory of fuzzy sets as an extraordinary discipline for the research communality. With the initiation of this theory, people started to welcome how uncertainty originating from human verdicts can manipulate scientific problems. Throughout the preceding decades, fuzzy logic has been fruitfully second-handed in functioning with abundant realistic applications. Some researchers have meaningfully pointed out that this fuzzy uncertainty is due to the incorrectness of the data measuring process and consequently in the stochastic sense it doesn't govern the modeling of unknowns in a system any longer. To consider the system from a further pragmatic perspective, one has to feature together stochastic uncertainty and fuzziness.

In numerous real-life situations, we evaluate descriptions of two objects realizing the procedure of such evaluation to be an imperative feature and there are numerous models to assess the difference involving objects. Along similar lines, the distances between fuzzy sets are as well significant for countless realistic applications. The depiction of the extent of dissimilarity involving two fuzzy subsets using a real number has been anticipated earlier by many researchers and it seems to be a constructive effort in several situations. Nevertheless, the prerequisite of conveying a particular number may escort us to the loss of vital information concerning this difference. Thus, there seems to be a prerequisite of divergence measures to discriminate whether the differences between two fuzzy subsets are in low or high membership degrees.

Provoked by the divergence models in probability spaces, Kapur [8] explained the perception of distance models in fuzzy spaces, and Bhandari and Pal [1] primarily outlined the divergence model for such spaces analogous to Kullback-Leibler [10] divergence model in probability spaces. Analogous to Renyi's [17], Havrada and Charvat's [5], and Sharma and Taneja's [19] probabilistic divergence models, Kapur [8] engendered a family of expressions concerning divergence models in discrete fuzzy spaces.

To enrich the literature on distance models in such spaces, Parkash [12] established the subsequent generalized model of fuzzy:

$$ {}_1D_\alpha^\beta(A,B)=[(\alpha-1)\,\beta]^{-1}\sum_{i=1}^{n}\left[\left\{\mu_A{}^{\alpha}(x_i)\mu_B{}^{1-\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}(1-\mu_B(x_i))^{1-\alpha}\right\}^{\beta}-1\right]; $$

$$ \alpha\neq 1,\alpha>0,\beta\neq 0 \qquad (1.1) $$

Parkash and Kumar [13, 14, 15] investigated and engendered certain fuzzy divergence models to make available their applications in a multiplicity of disciplines. Furthermore, Parkash and Sharma [16] stimulated the subsequent mathematical expression for the fuzzy divergence model analogous to Ferrari's [4] discrete divergence model:

$$ D^\alpha(A,B)=\frac{1}{\alpha}\sum_{i=1}^{n}\left[\left(1+\alpha\mu_A(x_i)\right)\log\frac{1+\alpha\mu_A(x_i)}{1+\alpha\mu_B(x_i)}+\left\{1+\alpha\left(1-\mu_A(x_i)\right)\right\}\log\frac{1+\alpha\left(1-\mu_A(x_i)\right)}{1+\alpha\left(1-\mu_B(x_i)\right)}\right],\alpha>0 \qquad (1.2) $$

The additional pioneer who contributed to fuzzy divergence models includes Bhandari et al. [2], Rosenfeld [18], Fan, Ma and Xie [3], Kobza [9], Joshi and Kumar [6], Joshi and Kumar [7], Markechová et al. [11], etc. In section 2, we have advocated a few innovative fuzzy divergence models and endowed with the learning of their imperative properties.

2. DISTANCE MODELS FOR DISCRETE FUZZY SPACES

In this segment, we advocate original generalized distance models in fuzzy spaces given by the subsequent mathematical expressions:

I. We firstly introduce a non-parametric fuzzy divergence model given by

$$ D_1(A:B)=2\left[n-2\sum_{i=1}^{n}\left\{\frac{\mu_A(x_i)\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)}+\frac{\{1-\mu_A(x_i)\}\{1-\mu_B(x_i)\}}{2-\mu_A(x_i)-\mu_B(x_i)}\right\}\right] \qquad (2.1) $$

The measure (2.1) corresponds to the Triangular Discrimination Measure given by

$$ \Delta(P;Q)=\sum_{i=1}^{n}\frac{\left(p_i-q_i\right)^2}{p_i+q_i} $$

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or $\Delta\left(P;Q\right)=2\left[1-W(P:Q)\right] \text{ where } W(P:Q)=2\sum_{i=1}^{n}\frac{p_i q_i}{p_i+q_i}$

To confirm the legitimacy of the anticipated model (2.1), we carry on subsequently:

For $\mu_A(x_i)=\mu_B(x_i)$, we acquire $D_1(A:B)=0$

Convexity: By straightforward computations, we acquire the subsequent expression:

$$\frac{d^2 D_1(A:B)}{d\mu_A^2(x_i)}=8\left[\frac{\mu_B^2(x_i)}{\left\{\mu_A(x_i)+\mu_B(x_i)\right\}^3}+\frac{\left\{1-\mu_B(x_i)\right\}^2}{\left\{2-\mu_A(x_i)-\mu_B(x_i)\right\}^3}\right]>0$$

which establishes the convexity of $D_1(A:B)$ fuzzy values $\mu_A(x_i)$.

Again, by uncomplicated working out, we acquire the subsequent expression:

$$\frac{d^2 D_1(A:B)}{d\mu_B^2(x_i)}=8\left[\frac{\mu_A^2(x_i)}{\left\{\mu_A(x_i)+\mu_B(x_i)\right\}^3}+\frac{\left\{1-\mu_A(x_i)\right\}^2}{\left\{2-\mu_A(x_i)-\mu_B(x_i)\right\}^3}\right]>0$$

which establishes the convexity of $D_1(A:B)$ fuzzy values $\mu_B(x_i)$.

Next, we will find the extremum of $D_1(A:B)$ a subject to the constraint $\sum_{i=1}^{n}\mu_A(x_i)=\alpha_0$. Let us think about the Lagrangian given by

$$L\equiv D_1(A:B)+\lambda\left(\sum_{i=1}^{n}\mu_A(x_i)-\alpha_0\right)$$

Accordingly $\frac{\partial L}{\partial\mu_A(x_i)}=0$ claims the subsequent appearance:

$$\left[\frac{\mu_B^2(x_i)}{\left\{\mu_A(x_i)+\mu_B(x_i)\right\}^2}-\frac{\left\{1-\mu_B(x_i)\right\}^2}{\left\{2-\mu_A(x_i)-\mu_B(x_i)\right\}^2}\right]=\frac{\lambda}{4}\quad\forall i$$

which is achievable only if $\mu_A(x_i)=\mu_B(x_i)\ \ \forall i$ and this gives $Min.\, D_1(A:B)=0$.

Similarly $\frac{dD_1(A:B)}{d\mu_B(x_i)}=0$ gives $Min.\, D_1(A:B)=0$.

Accordingly, we have the subsequent fundamental properties:

1. $D_1(A:B)\geq 0$
2. $D_1(A:B)=0$ iff $A=B$
3. $D_1(A:B)$ is unchangeable upon replacing $\mu_A(x_i)$ by $1-\mu_A(x_i)$ and $\mu_B(x_i)$ by $1-\mu_B(x_i)$
4. $D_1(A:B)$ is a convex function of $\mu_A(x_i)$ and $\mu_B(x_i)$

Under these conditions, the fuzzy divergence model proposed in (2.1) is an acceptable measure of directed divergence.

II. We next create another non-parametric fuzzy divergence model given by

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$$D_2(A:B)=\frac{1}{2}\sum_{i=1}^{n}\left[\begin{array}{l}\left\{\mu_A(x_i)\log\frac{2\mu_A(x_i)}{\mu_A(x_i)+\mu_B(x_i)}+\{1-\mu_A(x_i)\}\log\frac{2\{1-\mu_A(x_i)\}}{2-\mu_A(x_i)-\mu_B(x_i)}\right\}\\ +\left\{\mu_B(x_i)\log\frac{2\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)}+\{1-\mu_B(x_i)\}\log\frac{2\{1-\mu_B(x_i)\}}{2-\mu_A(x_i)-\mu_B(x_i)}\right\}\end{array}\right] \tag{2.2}$$

The measure (2.2) corresponds to Sibson [20] measure of divergence also called Jensen-Shannon model deliberated by the subsequent manifestation:

$$I(P;Q)=\frac{1}{2}\left[\sum_{i=1}^{n}p_i\ln\left(\frac{2p_i}{p_i+q_i}\right)+\sum_{i=1}^{n}q_i\ln\left(\frac{2q_i}{p_i+q_i}\right)\right].$$

To check the strength of the well-planned model (2.2), we continue subsequently:

For $\mu_A(x_i)=\mu_B(x_i)$, we acquire $D_2(A:B)=0$

Convexity: By uncomplicated working out, we obtain the subsequent quantitative expressions:

$$\frac{d^2D(A:B)}{d\mu_A^2(x_i)}=\frac{1}{2}\left[\frac{\mu_B(x_i)}{\mu_A(x_i)\{\mu_A(x_i)+\mu_B(x_i)\}}+\frac{1-\mu_B(x_i)}{\{1-\mu_A(x_i)\}\{2-\mu_A(x_i)-\mu_B(x_i)\}}\right]>0$$

and

$$\frac{d^2D(A:B)}{d\mu_B{}^2(x_i)}=\frac{1}{2}\left[\frac{\mu_A(x_i)}{\mu_B(x_i)\{\mu_A(x_i)+\mu_B(x_i)\}}+\frac{1-\mu_A(x_i)}{\{1-\mu_B(x_i)\}\{2-\mu_A(x_i)-\mu_B(x_i)\}}\right]>0$$

which verifies the convexity of the divergence model $D_2(A:B)$.

Next, we will find the extremum of a $D_2(A:B)$ subject to the constraint $\sum_{i=1}^{n}\mu_A(x_i)=\alpha_0$. Let us reflect on the subsequent Lagrangian:

$$L\equiv D_2(A:B)-\lambda\left(\sum_{i=1}^{n}\mu_A(x_i)-\alpha_0\right)$$

As a consequence $\frac{\partial L}{\partial\mu_A(x_i)}=0$, we assert the subsequent manifestation:

$$\frac{1}{2}\left[\log\frac{2\mu_A(x_i)}{\mu_A(x_i)+\mu_B(x_i)}-\log\frac{2\{1-\mu_A(x_i)\}}{2-\mu_A(x_i)-\mu_B(x_i)}\right]=\lambda\ \forall i$$

which is feasible iff $\mu_A(x_i)=\mu_B(x_i)\ \ \forall i$ and this gives $Min.\,D_2(A:B)=0$.

Similarly $\frac{dD_2(A:B)}{d\mu_B(x_i)}=0$ gives $Min.\,D_2(A:B)=0$.

As an outcome of the above decisive properties $D_2(A:B),$, of we argue that the discrete model wrought above is an acceptable divergence model.

III. We next generate a parametric fuzzy divergence model given by

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$$D^{\alpha}(A:B)=\frac{1}{\alpha(\alpha-1)}\sum_{i=1}^{n}\left[\begin{array}{l}\mu_B(x_i)\left\{\left\{\frac{\mu_A(x_i)}{\mu_B(x_i)}\right\}^{\alpha}-\alpha\left\{\frac{\mu_A(x_i)}{\mu_B(x_i)}\right\}+\alpha-1\right\}\\ \\ +\{1-\mu_B(x_i)\}\left\{\left\{\frac{1-\mu_A(x_i)}{1-\mu_B(x_i)}\right\}^{\alpha}-\alpha\left\{\frac{1-\mu_A(x_i)}{1-\mu_B(x_i)}\right\}+\alpha-1\right\}\end{array}\right]; \qquad (2.3)$$

$$\alpha>0,\alpha\neq 1$$

We monitor that in the limiting case, $D^{\alpha}(A:B)$ gets condensed to Bhandari and Pal's [1] model. Hence, we observe that the model projected in (2.3) is a generalized fuzzy divergence model.
Proceeding as above, we have proved the subsequent indispensable properties:

1. $D^{\alpha}(A:B)\geq 0$
2. $D^{\alpha}(A:B)=0$ iff $A=B$
3. $D^{\alpha}(A:B)$ is unchangeable upon replacing $\mu_A(x_i)$ by $1-\mu_A(x_i)$ and $\mu_B(x_i)$ by $1-\mu_B(x_i)$.
4. $D^{\alpha}(A:B)$ is convex.

Additionally, we have presented $D^{\alpha}(A:B)$ by employing diverse fuzzy values $\mu_A(x_i)$ as exposed in Table- 2.1 and consequently achieved Fig.-2.1 which provides evidence in favor of the convexity of the function $D^{\alpha}(A:B)$.

Table-2.1

$\mu_A(x_i)$	α	$D^{\alpha}(A:B)$	α	$D^{\alpha}(A:B)$	α	$D^{\alpha}(A:B)$	α	$D^{\alpha}(A:B)$
0.1		2.506244		3.047466		3.71413049		4.537404
0.2		0.99072		1.140744		1.31300349		1.511247
0.3		0.342301		0.378727		0.41795652		0.460288
0.4		0.0736		0.079493		0.08551806		0.091683
0.5	2.5	0	2.75	0	3.00	0	3.25	0
0.6		0.0736		0.079493		0.08551806		0.091683
0.7		0.342301		0.378727		0.41795652		0.460288
0.8		0.99072		1.140744		1.31300349		1.511247
0.9		2.506244		3.047466		3.71413049		4.537404

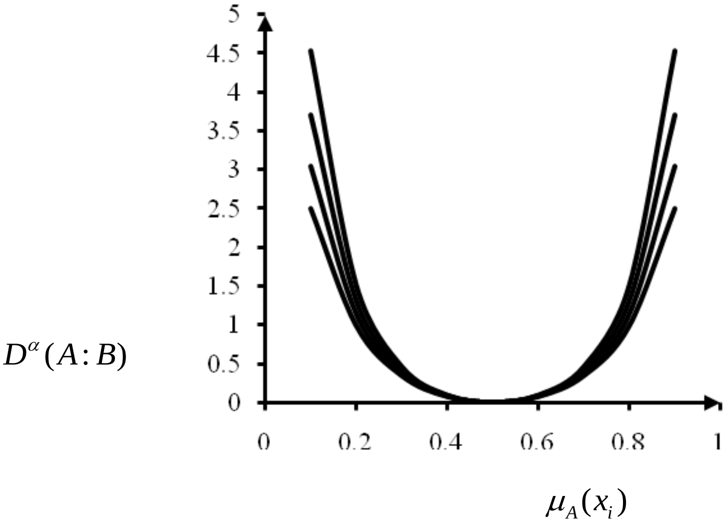

Fig.-2.1

Under these circumstances, the discrete model created in (2.3) is an accurate divergence model in the domain of fuzzy spaces.

Concluding remarks: It has been observed that where distance measures in probability spaces play a fundamental responsibility towards optimization problems in numerous disciplines of science and engineering, the divergence measures in fuzzy spaces do not lag behind. With this objective, we have advocated a family of original divergence models for discrete fuzzy distributions analogous to the surviving probabilistic models and studied their comprehensive properties for proving their legitimacy. Such developments in generating pioneering divergence models can be made comprehensive to surviving continuous distributions in fuzzy spaces.

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