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Study Of the Contractive fixed Points in G Metric Space

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ABSTRACT

The article contains the introduction, preliminary notions and definitions of G metric space are presented. Some fixed points are obtained for some self maps satisfying several contractive types.
Keywords: G metric space, fixed point, self-maps, G-contractive or reverse fixed point.

Introduction:
Let X be a non-empty set. Then $f: X \to X$ is aself-map on X. Then $x \in X$ is defined as fixed point of the function f, if $fa = a$. A set of conditions under which a map has the fixed point is said to be as fixed point theorem. The notion of the G-metric space, was initiated by Mustafa and sims [3] in an attempt to generalize fixed point theorems for own-maps on the metric space.

Preliminaries:
Mustafa and sims [3] introduced the notion of G-metric space as follows:
Definition 1: Let X be a nonempty set and $G: X \times X \times X \to [0, \infty)$ such that

(1) $G(a, b, c) = 0$ where $a, b, c \in X$ are such that a$= b = c$.
(2) $G(a, a, b) > 0$ for each$a, b \in X$ with $a \neq b$.
(3) $G(a, a, b) \leq G(a, b, a)$ for each$a, b, c \in X$ with $c \neq b$.
(4) $G(a, b, c) = G(\pi(a, b, c))$ for each$a, b, c \in X$ where $\pi(a, b, c)$ is a transformation on the set $\{a, b, c\}$.
(5) $G(a, b, c) \leq G(a, w, w) + G(w, b, c)$ for each$a, b, c, w \in X$.

Now G is defined to be as G-metric on X, and (X, G) denotes a Gmetric space. The elements of a Gmetric space are called its points.
Now T is a own map on the metric space (X, d) and $a_0 \in X$. The orbit $O_T(a_0) = < a_0, Ta_0, \ldots, T^n a_0, \ldots >$. The fixed point u in T is a reverse fixed point, if for every $O_T(a_0)$ converges to u. All the convergent sequence present in the metric space will have one and only one limit. So a reverse fixed point should also have a one and only one fixed point. Therefore it is clear that the exclusive fixed point for a own map will not be a contractive fixed point considered for it. The reality of the reverse fixed points for the own mapping in the metric spaces was analyzed by Edelstein [1], Reich [8], Leader and Hoyle [2] and Phaneendraand saravanan [5-7].

Definition 2:
The fixed point is considered as u of T on the Gmetric space (X, G) is a G reverse fixed point if the detourarrangement$a_0, Ta_0, \ldots, T^n a_0, \ldots$ at each $a_0 \in X$ is Gconvergent with bound$p$.

Theorem 1: Mohanta [4]
If(X, G) is considered to be the complete Gmetric space and$T$is considered to be the own map on Xsustaining
$G(Ta, Tb, Tc) \leq kmax\{G(a, Tb, Tb) + G(b, Ta, Ta) + G(c, Tc, Tc), G(b, Tc, Tc) + G(c, Tb, Tb) +$
$G(a, Ta, Ta), G(c, Ta, Ta) + G(a, Tc, Tc) + G(b, Tb, Tb)\}$ (1)
for all $a, b, c \in X$, where $0 < k < \frac{1}{3}$. Then T will have theexceptional fixed point u.

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Theorem 2:
If $(X, G)$is considered to be the comprehensive Gmetric space and T is considered to be the own map on X satisfying
$G(Ta, Tb, Tc) \leq \mu \ max\{G(a, Tb, Tb), G(b, Ta, Ta), G(c, Tc, Tc), G(b, Tc, Tc),$
$G(c, Tb, Tb), G(a, Ta, Ta), G(c, Ta, Ta), G(a, Tc, Tc), G(b, Tb, Tb)\}$ (2)
for all $a, b, c \in X$, where $\mu = 3k$ and $0 < k < \frac{1}{9}$.Then T will have the exceptional fixed point u.

Theorem 3: Vats[9]
If(X, G) isthecomprehensive Gmetric space and if$T$ is considered to be the own map on X satisfying
$G(Ta, Tb, Tc) \leq kmax\{G(a, Tb, Tb) + G(b, Tb, Tb) + G(c, Tc, Tc), G(a, Tb, Tb) + G(b, Ta, Ta) +$
$G(c, Tb, Tb), G(a, Tc, Tc) + G(b, Tc, Tc) + G(c, Ta, Ta)\}$ (3)
for all $a, b, c \in X$, where $0 < k < \frac{1}{4}$.Then T will have the exceptional fixed point u.

Main results:
Theorem 4:
If$(X, G)$is the comprehensive Gmetric space and if$T$ is considered to be the own map on G satisfying
$G(Ta, Tb, Tc) \leq \alpha G(a, b, c) + \beta \max\{G(a, Ta, Ta), G(b, Tb, Tb), G(c, Tc, Tc)\}$ (4)
for all $a, b, c \in X$ where $\alpha \geq 0, \beta \geq 0$, not both zero with $\alpha + \beta < 1$. Then u will be the G reverse fixed point of T.
Proof:
Let $a = T^{n-1}a_0,\ b = c = u$ in (4) we get

$$G(T^n a_0, u, u) = G(TT^{n-1}a_0, Tu, Tu)$$
$$\leq rG(T^{n-1}a_0, u, u) + s \max\{G(T^{n-1}a_0, T^n a_0, T^n a_0), G(u, Tu, Tu), G(u, Tu, Tu)\}$$
$$= rG(T^{n-1}a_0, u, u) + sG(G(T^{n-1}a_0, T^n a_0, T^n a_0) \quad (5)$$

Now with $a = T^{n-2}a_0, b = c = T^{n-1}a_0$ in (4) we get

$$G(T^{n-1}a_0, T^n a, T^n a_0)$$
$$\leq rG(T^{n-2}a_0, T^{n-1}a_0, T^{n-1}a_0)$$
$$+ s \max\{G(T^{n-2}a_0, T^{n-1}a_0, T^{n-1}a_0), G(T^{n-1}a_0, T^n a_0, T^n a_0), G(T^{n-1}a_0, T^n a_0, T^n a_0)\}$$
$$= rG(T^{n-2}a_0, T^{n-1}a_0, T^{n-1}a_0) + sN \quad (6)$$

Where $N = \max\{G(T^{n-2}a_0, T^{n-1}a_0, T^{n-1}a_0)G(T^{n-1}a_0, T^n a_0, T^n a_0)\}$.
If $N = G(T^{n-1}a_0, T^n a_0, T^n a_0)$. Then (6) becomes

$$G(T^{n-1}a_0, T^n a, T^n a_0) \leq rG(T^{n-2}a_0, T^{n-1}a_0, T^{n-1}a_0) + sG(T^{n-1}a_0, T^n a_0, T^n a_0)$$
$$G(T^{n-1}a_0, T^n a, T^n a_0) \leq \frac{r}{(1-s)} G(T^{n-2}a_0, T^{n-1}a_0, T^{n-1}a_0)$$
$$G(T^{n-1}a_0, T^n a, T^n a_0) \leq (\tfrac{r}{(1-s)})^n\ G(a_0, Ta_0, Ta_0) \quad (7)$$

Hence by induction

$$G(T^{n-1}a_0, T^n a, T^n a_0) \leq r^{n-1}G(Ta_0, u, u) + s(1 + r + r^2 + \cdots + r^{n-2})(\frac{r}{(1-s)})^{n-1}\ G(a_0, Ta_0, Ta_0)$$
$$\leq r^{n-1}G(Ta_0, u, u) + (\tfrac{s}{1-r})(\tfrac{r}{(1-s)})^{n-1}\ G(a_0, Ta_0, Ta_0) \quad (8)$$

Suppose that $N =\ G(T^{n-2}a_0, T^{n-1}a_0, T^{n-1}a_0)$, then (6) becomes

$$G(T^{n-1}a_0, T^n a, T^n a_0) \leq rG(T^{n-2}a_0, T^{n-1}a_0, T^{n-1}a_0) + sG(T^{n-2}a_0, T^{n-1}a_0, T^{n-1}a_0)$$
$$= (r + s)G(T^{n-2}a_0, T^{n-1}a_0, T^{n-1}a_0)$$

Hence

$$G(T^{n-1}a_0, T^n a, T^n a_0) \leq (r + s)^{n-1}G(a_0, Ta_0, Ta_0)$$

Substituting in (5) we get

$$G(T^n a_0, u, u) \leq rG(T^{n-1}a_0, u, u) + s(r + s)^{n-1}G(a_0, Ta_0, Ta_0)$$

By induction

$$G(T^{n-1}a_0, T^n a, T^n a_0)$$
$$\leq r^{n-1}G(Ta_0, u, u) + s(1 + r + r^2 + \cdots + r^{n-2})\ (r + s)^{n-1}G(a_0, Ta_0, Ta_0)$$
$$\leq r^{n-1}G(Ta_0, u, u) + \tfrac{s}{1-r}(r + s)^{n-1}G(a_0, Ta_0, Ta_0) \quad (5.9)$$

Taking lim$n \to \infty$ in (5.8) and (5,9)

$$G(T^n a_0, u, u) \to 0$$

Hence $T^n a_0 \to u$ for every $a_0 \in X$.
Hence the theorem is proved.
Theorem 5:

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If u is considered to be the fixed point of the map$T$sustaining (1) where $0 < k < \frac{1}{3}$. Then u be the G reverse fixed point of T.

Proof:

Let $a = T^{n-1}a$ and $b = c = u$ in (1) using the condition

$G(a, b, c) \leq G(x, w, w) + G(w, b, c)$ for all $a, b, c, w \in X$, then we get

$$G(T^n a, u, u) = G(T^n a, Tu, Tu)$$

$$\leq kmax\{G(T^{n-1}a, Tu, Tu) + G(u, T^n a, T^n a) + G(u, Tu, Tu), G(u, Tu, Tu) + G(u, Tu, Tu) + G(T^{n-1}a, T^n a, T^n a), G(u, T^n a, T^n a) + G(T^{n-1}a, Tu, Tu) + G(u, Tu, Tu)\}$$

$$\leq kmax\{G(T^{n-1}a, u, u) + G(u, T^n a, T^n a), G(T^{n-1}a, T^n a, T^n a), G(u, T^n a, T^n a) + G(T^{n-1}a, u, u)$$

$$\leq k[G(u, T^n a, T^n a) + G(T^{n-1}a, u, u)]$$

$$\leq k[2G(T^n a, u, u) + G(T^{n-1}a, u, u)]$$

$$\Rightarrow (1 - 2k\ G(T^n a, u, u) \leq kG(T^{n-1}a, u, u)$$

$$\Rightarrow G(T^n a, u, u) \leq \frac{k}{1-2k} G(T^{n-1}a, u, u)$$

Since $\frac{k}{1-2k} < 1$, then $G(T^n a, u, u) \to 0$ as $n \to \infty$ for each value of $a \in X$.

Now$u$ will be a G reverse fixed point of T.

Hence the theorem is proved.

Theorem 6:

Ifu is considered to be the fixed point of the map$T$sustaining (3) here $0 < k < \frac{1}{4}$. Then u is a G reverse fixed point of T.

Proof:

Let $b = c = u$ in (3) and using the condition

$G(a, b, c) \leq G(x, w, w) + G(w, b, c)$ for all $a, b, c, w \in X$, then we get

$$G(T^n a, u, u) = G(T^n a, Tu, Tu)$$

$$\leq kmax\{G(T^{n-1}a, T^n a, T^n a) + G(T^{n-1}a, Tu, Tu) + G(T^{n-1}a, Tu, Tu), G(u, Tu, Tu) + G(u, T^n a, T^n a) + G(u, Tu, Tu), G(u, Tu, Tu) + G(u, T^n a, T^n a) + G(u, Tu, Tu)\}$$

$$= kmax\{G(T^{n-1}a, T^n a, T^n a) + 2G(T^{n-1}a, u, u), G(u, T^n a, T^n a)\}$$

$$= kmax\{G(u, T^n a, T^n a) + G(T^{n-1}a, u, u) + 2G(T^{n-1}a, u, u), G(u, T^n a, T^n a)\}$$

$$= k\{G(u, T^n a, T^n a) + 3G(T^{n-1}a, u, u)\}$$

$$\leq \{2G(T^n a, u, u) + 3G(T^{n-1}a, u, u)\}$$

$$\Rightarrow G(T^n a, u, u) \leq \frac{3k}{1-2k} G(T^{n-1}a, u, u)$$

By induction

$$\Rightarrow G(T^n a, u, u) \leq \left(\frac{3k}{1-2k}\right)^n G(Ta, u, u)$$

When $\to \infty$, $T^n a \to u$ for every $a \in X$.

Here$u$ will be the G reverse fixed point.

Theorem 7:

If(X, G) is considered to be the comprehensive Gmetric space and the map T is considerd to be the ownmap on G sustaining

$$G(Ta, Tb, Tc) \leq k \max\{G(a, Ta, Ta), G(b, Tb, Tb), G(c, Tc, Tc), G(a, Tb, Tb), G(b, Tc, Tc), G(c, Ta, Ta), G(a, Tc, Tc), G(b, Ta, Ta), G(c, Tb, Tb), G(a, Tb, Tc), G(b, Tc, Ta, G(c, Ta, Tb), G(a, b, Tc), G(b.c, Ta), G(c, a, Tb), G(a, b, c)\} \quad (10)$$

for all $a, b, c \in X$, where $0 < k < \frac{1}{3}$. Then u will be the G reverse fixed point of T.

Proof:

Let $a = T^{n-1}a$ and $b = c = u$ in (10), using the condition

$G(a, b, c) \leq G(x, w, w) + G(w, b, c)$ for all $a, b, c, w \in X$, then we have

$$G(T^n a, u, u) = G(T^n a, Tu, Tu)$$

$$\leq kmax\{G(T^{n-1}a, T^n a, T^n a), G(u, Tu, Tu), G(u, Tu, Tu), G(T^{n-1}a, Ta, Ta), G(u, Tu, Tu), G(u, T^n a, T^n a), G(T^{n-1}a, Tu, Tu), G(u, T^n a, T^n a), G(u, Tu, Tu), G(T^{n-1}a, Tu, Tu), G(u, Tu, T^n a), G(u, T^n a, Tu), G(T^n a, u, Tu), G(u, u, T^n a), G(u, T^n a, Tu), G(T^n a, u, u)\}$$

$$\leq kmax\{G(T^{n-1}a, T^n a, T^n a), 0, 0, G(T^{n-1}a, u, u), 0, G(u, T^n a, T^n a), G(T^{n-1}a, u, u), G(u, T^n a, T^n a), 0, G(T^{n-1}a, u, u), G(u, u, T^n a), G(u, T^n a, u), G(T^n a, u, u), G(u, u, T^n a), G(u, T^n a, u), G(T^n a, u, u)\}$$

$$\leq kmax\{G(T^{n-1}a, T^n a, T^n a), G(T^{n-1}a, u, u), G(u, T^n a, T^n a), G(u, u, T^n a)\}$$

$$\leq kmax\{\, G(T^{n-1}a,u,u) + \, G(u,T^na,T^na), G(u,u,T^na)\}$$
$$\leq kmax\{\, G(T^{n-1}a,u,u) + \, 2G(u,u,T^na), G(u,u,T^na)\}$$
$$= k[G(T^{n-1}a,u,u) + 2G(u,u,T^na)$$

So that

$$\Rightarrow G(T^na,u,u) \leq \frac{k}{1-2k} G(T^{n-1}a,u,u)$$

Since $\frac{k}{1-2k} < 1$, we get $G(T^na,u,u) \to 0$ as $n \to \infty$for every $a \in X$.
Hence u will be the G reverse fixed point of T.

Conclusion:

New generalized fixed point theorem in a Gmetric space is obtained by employing a broader inequality which generalize the results obtained by researchers such as Mohanta and Vats. G –reversefixed points are attained for the several contraction types.

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