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A Study On Convergence Analysis Of Runge-Kutta Fehlberg Method To Solve Fuzzy Delay Differential Equations

[1]Ch. Subba Reddy, [2*]T.L.Yookesh, [3]E. Boopathi Kumar
[1,2] Department of Mathematics, Vignan's Foundation for Science, Technology & Research, Guntur-13, INDIA.
[3] Department of Information Technology, Bharathiar University, Coimbatore – 46, INDIA.

E.Mail: *[1]chreddy12@gmail.com, [2*]renu_yookesh@yahoo.co.in & [3]edboopathikumar@gmail.com*

ABSTARCT

Here the converegence of Runge-Kutta-Fehlberg method (RKF) is studied for solving the Delay Differential Equations (DDE) under the fuzzy concepts, since the fuzzy helps in resolving the uncertinanity in the results. The numerical results are examined and shows the effeciency of this method.

Key Words: Fuzzy numerical analysis, time delay, Iterative technique.

I. INTRODUCTION

The idea of timing with different conditions has been the subject of more applied design research. Thus, it is imperative to promote digital models and digital frameworks that can be used to solve regularly used questionable conditions. The idea of a fuzzy derivative was dreamed up by Chang and Zadeh [7]. Then, Dubois and Prade [8] introduced the idea of fuzzy derivative in relation to the augmentation rule. The expression "Fuzzy differential equation" was introduce by Kandel and Byat [13]. Abbasbandi and Allahviranloo [2] performed mathematical calculations to handle the fuzzy differential conditions with respect to the Seikkala derivative [19]. Jafari et al. [10] fail to satisfy the nth requirement of the fuzzy differential conditions by variational iteration method. Allahviranloo et al. [1, 3] used the correct indicator strategy to track the mathematical arrangement of the differential equation with fuzzy. Driver [9] has written a book on general differential conditions and differential delay conditions which explains the conditions explicitly. Bellen et al. [5, 6] presented mathematical answers for DDE. Khastan et al., in [14], the ideas of differential conditions for fuzzy change under summed differentiability have been discussed. Barzinji et al [4] focused on the linear differential differential frames to analyze the security of a coherent state.

The delay differential equation general form of order one is,

$$\begin{aligned} g'(\mathrm{h}) &= z\left(t, g(\mathrm{h}), g_{\mathrm{n}}(\mathrm{h}-\mathrm{r}_{\mathrm{i}})\right), h \ge h_0,\ i = 1,2,3,\ldots,\mathrm{n}, \\ g(\mathrm{h}) &= \phi(\mathrm{h}), h \ge h_0 \end{aligned} \tag{1.1}$$

Abbasbandi and Allahwiranloo [2] introduced a numerical technique to tune fuzzy differential conditions using the Runge-Kutta methodology for the fourth survey. Pederson and Sambandham [18] focused on the numerical assignment of fuzzy differential delay circumstances using the Runge-Kutta method. Al-Rawi et al. [21] discussed a numerical technique for solving the differential carry constraints using the fourth-order Runge-Kutta strategy. K.Kanagarajan et al. [12, 11] eliminated Runge-Kutta-Nystrom and the 5^{th}-Runge-Kutta strategy for treating fuzzy-delay differential equations. V. Parimala et al.[17] used the 2^{nd}-Runge-Kutta technique to solve fuzzy differential circumstances with fuzzy initial circumstances. Narayanamoorthy et al. [15, 16] used Runge-Kutta's third technique to resolve fuzzy differential circumstances using Seikkal derivative [19].

This article is an extension of [16] and this article aims to see how this time lag differential condition method works in fuzzy environment. The desired strategy quickly fits into the framework of the concrete development. We describe a mathematical guide to understand the proposed strategy.

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II. Preliminaries

Definition 1. A fuzzy set "$\bar{A}$" defined on "X" and a number $\gamma \in [0,1]$, the γ-cut is "$\bar{A}^{\gamma}$" and the strong γ - *cut* is "$\bar{A}^{\gamma+}$" are the crisp sets $\bar{A}^{\gamma} = \{x \mid A(x) \geq \gamma\}$; $\bar{A}^{\gamma+} = \{x \mid A(x) > \gamma\}$.

Definition 2. Fuzzy set "$\bar{A}$" is the triangular fuzzy number with peak (or center) "a", left width $\alpha > 0$ and right $\beta > 0$, has the following form

$$\mu(x) = \begin{cases} 1 - \dfrac{a-x}{\alpha} & if \quad a - \alpha < x < a, \\[2ex] 1 - \dfrac{x-a}{\beta} & if \quad a < x < a + \beta, \\[2ex] 0 & otherwise. \end{cases}$$

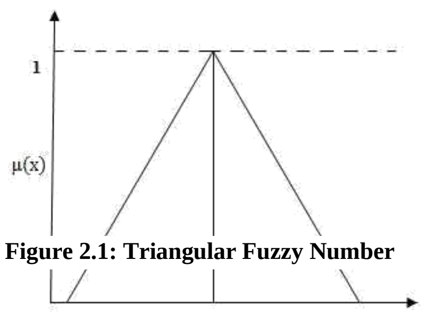

Figure 2.1: Triangular Fuzzy Number

Definition 3. For arbitrary fuzzy numbers $\psi = \left(\overline{\psi}, \underline{\psi}\right)$, $\vartheta = \left(\overline{\vartheta}, \underline{\vartheta}\right)$ the quality

$$D(\psi, \vartheta) = \left\{ \sup_{0 \leq \gamma \leq 1} \left| \underline{\psi}(\gamma) - \underline{\vartheta}(\gamma) \right|, \sup_{0 \leq \gamma \leq 1} \left| \overline{\psi}(\gamma) - \overline{\vartheta}(\gamma) \right| \right\}$$ is the distance between ψ *and* ϑ.

III. CONVEREGENCE OF FUZZY DELAY DIFFERENTIAL EQUATIONS

Assume the first order differential equation [16],

$$\begin{cases} g'(m) = p(m, g(m)), \; m \in [m_0, M] \\ g(m_0) = g_0 \end{cases}$$

where g is a crisp function of $m, p(m, g)$. We represent the fuzzy function as $g = \left[\underline{g}, \overline{g}\right]$. We have $\left[p(m, g)\right] = \left[\underline{p}(m, g), \overline{p}(m, g)\right]$ and $\underline{p}(m, g) = P\left[m, \underline{g}, \overline{g}\right]$; $\overline{p}(m, g) = H\left[m, \underline{g}, \overline{g}\right]$

By using the extension principle [16],

$$p(\, m, g(m)\,)\,(s) = \sup\{\, g(m)(\tau) \setminus s = p(m, \tau)\}, \;\; s \in R$$

From this it follows that $\left[p(m, g(m))\right]_{\gamma} = \left[\underline{p}(m, g(m); \gamma), \overline{p}(m, g(m); \gamma)\right], \;\; \gamma \in (0, 1]$

Where

$$\underline{p}(m, g(m); \gamma) = \min\left\{p(m, v) \setminus v \in \left[g(m)\right]_{\gamma}\right\}; \; \overline{p}(m, g(m); \gamma) = \max\left\{p(m, v) \setminus v \in \left[g(m)\right]_{\gamma}\right\} \quad (3.1)$$

Theorem 3.1.

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Let "p" satisfy $\left|p(d,\underline{v})-p(d,\bar{v})\right| \le h\left(d,\left|v-\bar{v}\right|\right)$, $d \ge 0, \underline{v}, \bar{v} \in \Re$ where $h:\Re \to \Re$ will be a continuous mapping, so that $\gamma \to h(d,\gamma)$ is increasing and the initial value problem $u'(d) = h(d,u(d),u(d-\tau))$, $u(0)=0.$ Has the result on $\Re$ for $u_0>0$ and that $u(d) = 0$ is the only way out of (3.1). Thus it has unique solution.

Lemma 3.2. Let the numbers in order $\left\{\omega_i\right\}_{i=0}^{I}$ satisfy $\left|\omega_{i+1}\right| \le G\left|\omega_i\right| + H, 0 \le i \le I-1,$

for some given constants G and H. Then $\left|\omega_i\right| \le G^i\left|\omega_0\right| + H\dfrac{G^{i-1}}{G-1}, 0 \le i \le I.$

Lemma3.3: Let the attangement of numbers $\left\{\omega_i\right\}_{i=0}^{I}$ and $\left\{\upsilon_i\right\}_{i=0}^{I}$ satisfy

$$\left|\omega_{i+1}\right| \le \left|\omega_i\right| + G\max\{\left|\omega_i\right|,\left|\upsilon_i\right|\} + H,$$
$$\left|\upsilon_{i+1}\right| \le \left|\upsilon_i\right| + G\max\{\left|\upsilon_i\right|,\left|\upsilon_i\right|\} + H$$

for some given positive constants G and H, then $V_i = \left|\omega_i\right| + \left|\upsilon_i\right|$ where $\bar{G}=1+2G$ and $\bar{H}=2H$.

IV. RUNGE-KUTTA-FEHLBERG METHOD

In this session, for solving the fuzzy delay differential equation the proposed method, Runge-Kutta-Fehlberg method (RKF) is constructed under fuzzy environment. Let $U=[U_1,U_2]$ be exact solution and $u=[u_1,u_2]$ is an approximate solution to the ambiguous initial value problem. The solution is the calculated phase points

$$\delta=\frac{D-d_0}{N},\ d_1=d_0+i\delta,\ 0\le i\le N$$

Then we obtain the,

$$\underline{u}(\kappa+1) = \min\left(u_\kappa+\frac{25}{216}\kappa_1+\frac{1408}{2565}\kappa_3+\frac{2197}{4104}\kappa_4-\frac{1}{5}\kappa_5\right);$$
$$\bar{u}(\kappa+1) = \max\left(u_\kappa+\frac{25}{216}\kappa_1+\frac{1408}{2565}\kappa_3+\frac{2197}{4104}\kappa_4-\frac{1}{5}\kappa_5\right)$$

where

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$$\kappa_1(d,\gamma) = \min \delta\left(f\left(d_\kappa, u_\kappa\right)\right);\ \max \delta\left(f\left(d_\kappa, u_\kappa\right)\right)$$

$$\kappa_2(d,\gamma)=\min \delta\left(f\left(d_\kappa+\frac{1}{4}\delta, u_\kappa+\frac{1}{4}\kappa_1\right)\right);\ \max \delta\left(f\left(d_\kappa+\frac{1}{4}\delta, u_\kappa+\frac{1}{4}\kappa_1\right)\right)$$

$$\kappa_3(d,\gamma)=\min \delta\left(f\left(d_\kappa+\frac{3}{8}\delta, u_\kappa+\frac{3}{32}\kappa_1+\frac{9}{32}\kappa_2\right)\right);\ \max \delta\left(f\left(d_\kappa+\frac{3}{8}\delta, u_\kappa+\frac{3}{32}\kappa_1+\frac{9}{32}\kappa_2\right)\right)$$

$$\kappa_4(d,\gamma)=\min \delta\left(f\left(d_\kappa+\frac{12}{13}\delta, u_\kappa+\frac{1932}{2197}\kappa_1-\frac{7200}{2197}\kappa_2+\frac{72916}{2197}\kappa_3\right)\right);$$

$$\max \delta\left(f\left(d_\kappa+\frac{12}{13}\delta, u_\kappa+\frac{1932}{2197}\kappa_1-\frac{7200}{2197}\kappa_2+\frac{72916}{2197}\kappa_3\right)\right)$$

$$\kappa_5(d,\gamma)=\min \delta\left(f\left(d_\kappa+\delta, u_\kappa+\frac{439}{216}\kappa_1-8\kappa_2+\frac{3680}{513}\kappa_3-\frac{845}{4104}\kappa_4\right)\right);$$

$$\max \delta\left(f\left(d_\kappa+\delta, u_\kappa+\frac{439}{216}\kappa_1-8\kappa_2+\frac{3680}{513}\kappa_3-\frac{845}{4104}\kappa_4\right)\right)$$

$$\kappa_6(d,\gamma)=\min \delta\left(f\left(d_\kappa+\frac{1}{2}\delta, u_\kappa-\frac{8}{27}\kappa_1+2\kappa_2-\frac{3544}{2565}\kappa_3+\frac{1859}{4104}\kappa_4-\frac{11}{40}\kappa_5\right)\right);$$

$$\max \delta\left(f\left(d_\kappa+\frac{1}{2}\delta, u_\kappa-\frac{8}{27}\kappa_1+2\kappa_2-\frac{3544}{2565}\kappa_3+\frac{1859}{4104}\kappa_4-\frac{11}{40}\kappa_5\right)\right)$$

observance argument the value of γ is fixed and then the exact and approximate solution of d_η are represented by ,

$$[U(d_\eta)]_\gamma = [U_1(d_\eta;\gamma), U_2(d_\eta;\gamma)]$$
$$[u(d_\eta)]_\gamma = [u_1(d_\eta;\gamma), u_2(d_\eta;\gamma)],\ 0 \le \eta \le N \tag{4.1}$$

By this, we finding $[u_1(d_\eta;\gamma), u_2(d_\eta;\gamma)]$ converges to $[U_1(d_\eta;\gamma), U_2(d_\eta;\gamma)]$ respectively at $\delta \to 0$.

Theorem 4.1.

Consider the systems (4.1), for $\gamma \in [0,1]$, then $\lim\limits_{f\to 0}\left[\underline{u}(\mathrm{d}_\eta)\right]_\gamma = \underline{U}\left(d_\eta;\gamma\right); \lim\limits_{f\to 0}\left[\overline{u}(\mathrm{d}_\eta)\right]_\gamma = \overline{U}\left(d_\eta;\gamma\right)$.

V. ILLUSTRATIVE EXAMPLE

Example 1: Consider the LFDDE,

$$u'(\chi) = \frac{1}{2} e^{\frac{\chi}{2}} u\left(\frac{\chi}{2}\right) + \frac{1}{2} u(\chi) \tag{5.1}$$

with $u(0) = (\gamma, 2-\gamma)$.

The exact solution of (5.1) is

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$$\underline{u}(\chi,\gamma)=\gamma e^{\chi}$$
$$\overline{u}(\chi,\gamma)=(2-\gamma)e^{\chi} \tag{5.2}$$

To approximate (5.1) by the RKF method, and general form as (5.3)

$$\underline{u}'(\chi,\gamma)=\frac{1}{2}e^{\frac{\chi}{2}}\underline{u}\left(\frac{\chi}{2}\right)+\frac{1}{2}\underline{u}(\chi)$$
$$\overline{u}'(\chi,\gamma)=\frac{1}{2}e^{\frac{x}{2}}\overline{u}\left(\frac{\chi}{2}\right)+\frac{1}{2}\overline{u}(\chi) \tag{5.3}$$

with the initial condition $\underline{u}(0)=\gamma,\ \overline{u}(0)=2-\gamma.$

The value of the approximate method is tabulated below with $\chi=1.$

Table 5.1: Approximate Values

	Runge-Kutta Fehlberg method		Runge-Kutta fourth order method	
γ	$\underline{u}(\chi,\gamma)$	$\overline{u}(\chi,\gamma)$	$\underline{u}(\chi,\gamma)$	$\overline{u}(\chi,\gamma)$
0	1.718802	3.718802	1.718282782	3.718282782
0.2	1.918802	3.518802	1.918282782	3.518282782
0.4	2.118802	3.318802	2.118282782	3.318282782
0.6	2.318802	3.118802	2.318282782	3.118282782
0.8	2.518802	2.918802	2.518282782	2.918282782
1	2.718802	2.718802	2.718282782	2.718282782

Table 5.2: Error Table

	Exact Solution		**Runge-Kutta Fehlberg method**		**Runge-Kutta fourth order method**	
γ	$\underline{U}(\chi,\gamma)$	$\overline{U}(\chi,\gamma)$	$\underline{u}(\chi,\gamma)$	$\overline{u}(\chi,\gamma)$	$\underline{u}(\chi,\gamma)$	$\overline{u}(\chi,\gamma)$
0	0	5.436563657	1.719012514	1.71755143	1.71828278	1.718280875
0.2	0.543656366	4.892907291	1.375356148	1.37389478	1.37462642	1.374624509
0.4	1.087312731	4.349250925	1.031699783	1.03023841	1.03097005	1.030968143
0.6	1.630969097	3.805594559	0.680932157	0.68658205	0.68731369	0.687311777
0.8	2.174625463	3.261938194	0.337275788	0.34292568	0.34365732	0.343655412
1	2.718281828	2.718281828	0.000730686	0.00073069	$9.54*10^{-7}$	$9.54*10^{-7}$

Examination of above results are compared in following figures, in figure 5.1 both the method are compared with the exact solution and in figure 5.2 shows the bond between the two method.

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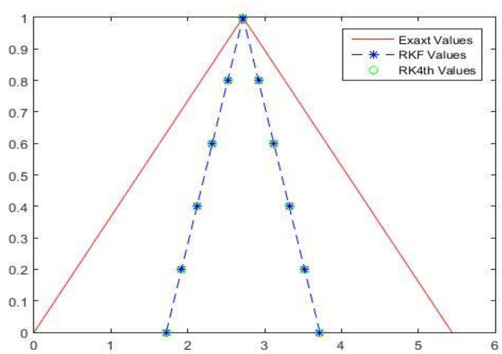

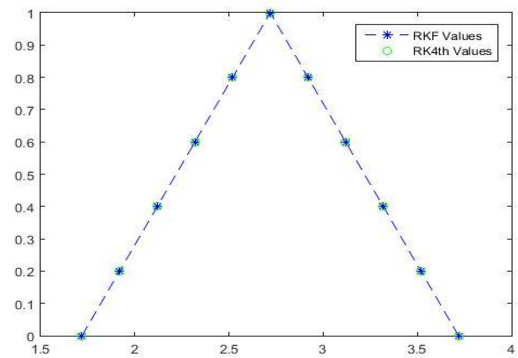

Figure 5.1: Comparison Between Approximate and Exact Solution **Figure 5.2: Comparision Between RKF and RK4th order Methods**

VI. Conclusion

In it, the integration of the proposed method to solve fuzzy delay difference equations is examined. A numerical example shows the performance of the implement number system and compares it with the exact and Runge-Kutta 4^{th}-order system, to show proposed method accuracy.

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