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On Polynomial Orthogonal of Type I Matrices

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ABSTRACT: The concept of polynomial orthogonal of type I matrices are introduced. We define the index of orthogonal of type I matrices and we extended some results of orthogonal of type I matrix to polynomial orthogonal of type I matrices.

KEYWORDS: Orthogonal matrix, orthogonal of type I matrix, determinant, inverse, transpose.

1. INTRODUCTION

Matrices provide a very powerful tool for dealing with linear models. In multidimensional system theory, problems related to multivariable control system invertibility require the use of generalized inverse of matrices whose elements are polynomials in several variables with coefficients over a real field (or) a rational field (or) an integral domain of integers. Orthogonal matrices are important for a number of reasons, both theatrical and practical. A polynomial matrix of degree *n*, $A(\lambda) = A_0 + A_1\lambda + A_2\lambda^2 + \dots\dots\dots + A_n\lambda^n$ is said to be polynomial matrix if all entries of $A(\lambda)$ are polynomials. Polynomials and polynomial matrices arise naturally as modelling tools in several areas of applied mathematics, sciences and engineering, especially in systems theory **[2, 3, 4, 5, 6, 7, 8]**. In this paper, we introduced the new type of matrices, we called it polynomial orthogonal of type I matrices and extended some results of polynomial orthogonal of type I matrices.

2. PRELIMINARIES

DEFINITION: 2.1[1]

A square matrix *A* is called an orthogonal of type I matrix if $A^k (A^T)^k = I_n$ *and* $(A^T)^k (A)^k = I_n$, for some $k \epsilon N$

DEFINITION: 2.2[1]

Let *A* be an orthogonal of type I matrix and There exists positive integer *k* with $A^k (A^T)^k = I_n$ is called the index of *A*. We say that *A* is an orthogonal of type I of period *k* .

3. Polynomial Orthogonal of Type I Matrices

DEFINITION: 3.1

A polynomial orthogonal of type I matrix is a polynomial orthogonal of type I matrix whose coefficient matrices are orthogonal of type I matrices.

That is, $A_i^{\,k}(A_i^{\,T})^k = I_n \; and \; (A_i^{\,T})^k (A_i)^k = I_n$, for some $k\epsilon N$

EXAMPLE: 3.2

Let $A(\lambda) = A_0 + A_1\lambda + A_2\lambda^2 + \dots\dots\dots + A_n\lambda^n$ be polynomial orthogonal of type I matrix. Here coefficient matrices $A_i's$ are orthogonal of type I matrices.

$$Let\ A(\lambda) = \begin{pmatrix} i - i\lambda & 0 \\ 0 & i - i\lambda \end{pmatrix} = A_0 + A_1\lambda$$

$$A(\lambda) = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}\lambda$$

where $A_0 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ and $A_1 = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$

Put k = 1, $A_0 A_0^{\,T} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

Put k = 2, $A_0^{\,2}(A_0^{\,T})^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Put k = 3, $A_0^{\,3}(A_0^{\,T})^3 = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}$

Put k = 4, $A_0^{\,4}(A_0^{\,T})^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Put k = 5, $A_0^{\,5}(A_0^{\,T})^5 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

Put k = 6, $A_0^{\,6}(A_0^{\,T})^6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The index of A_0 is 2, then $A_0^{\,k}(A_0^{\,T})^k = I_2 \; and$ k=2,4,6,….. and *A* is of period 2.

Similarly we can prove A_1 .

THEOREM: 3.3

If $A(\lambda)$ is a polynomial orthogonal of type I matrix. Whose coefficient matrices of $A(\lambda)$ are orthogonal of type I matrices. That is $A_0, A_1, A_2, \dots\dots\dots A_n$ are orthogonal of type I matrices. The matrix $A_i^{'}s$ are orthogonal of type I of index k if and only if A_i^m are orthogonal of type I matrices of index *k* for each $m\epsilon\mathbb{N}$

Proof:

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If $A(\lambda)$ is a polynomial orthogonal of type I matrix. Whose coefficient matrices of $A(\lambda)$ are orthogonal of type I matrices. That is $A_0, A_1, A_2, ...A_n$ are orthogonal of type I matrices.

Suppose that $A_i^{'}s$ are orthogonal of type I matrices.

To prove A_i^m are orthogonal of type I matrices.

So that , $A_i^{\,k}(A_i^{\,T})^k = I_n$ for some $k\epsilon R$

Taking m on both side

$$(A_i^{\,k}(A_i^{\,T})^k)^m = (I_n)^m \quad \text{for } m \in N$$

$$(A_i^{\,k})^m((A_i^{\,T})^k)^m = I_n$$

$$(A_i^{\,m})^k((A_i^{\,m})^T)^k = I_n$$

Hence A_i^m are orthogonal of type I matrices .

Conversely, suppose that A_i^m are orthogonal of type I matrices for each $m \in N$

To prove $A_i^{'}s$ are orthogonal of type I matrices of index k.

Since $(A_i^{\,m})^k((A_i^{\,m})^T)^k = I_n$, for $m \in N$,

Each of A_i^2 and A_i^3 are orthogonal of type I matrices of index k.

So,

$$I_n = (A^3)^k((A^3)^T)^k$$

$$I_n = A_i^{\,k}(A_i^{\,2})^k((A_i^{\,2})^T)^k(A_i^{\,T})^k \quad \text{for } A^2 \text{ is an orthogonal of type I matrix.}$$

$$I_n = A_i^{\,k}(A_i^{\,T})^k$$

Hence $A_i^{'}s$ are orthogonal of type I matrix of index k.

THEOREM: 3.4

If $A(\lambda)$ is an polynomial orthogonal of type I matrix is a polynomial orthogonal of type I matrix whose coefficient matrices are orthogonal of type I matrices. Then $\det A_i^k = \pm 1$.

Proof:

Let $A(\lambda) = A_0 + A_1\lambda + A_2\lambda^2 + ...+ A_n\lambda^n$ be polynomial orthogonal of type I matrix. Here coefficient matrices $A_i's$ are orthogonal of type I matrices.

Let *A* be an orthogonal of type I matrix of index *k*.

Given that , $A_i^{\,k}(A_i^{\,T})^k = I_n$

Taking det on both sides

$$\det(A_i^{\,k}(Ai^T)^k) = \det(I_n)$$

$$\det A_i^{\,k}.\det(A_i^{\,T})^k = \det(I_n)$$

Since, $\det A^k = \det(A^T)^k$

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ie) $\det A_i^{\,k} . \det A_i^{\,k} = 1$

$(\det A_i^{\,k})^2 = 1$

$\det(A_i^{\,k}) = \pm 1$

Hence $\det(A_i^{\,k}) = \pm 1$

EXAMPLE: 3.5

Let
$$A(\lambda) = \begin{pmatrix} i - i\lambda & 0 \\ 0 & i - i\lambda \end{pmatrix} = A_0 + A_1\lambda$$

$$A(\lambda) = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}\lambda$$

where $A_0 = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$ and $A_1 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$

$$\det A_0 = \left[i^2 - 0 \right] = -1$$

$$\det A_1 = \left[i^2 - 0 \right] = -1$$

Hence $\det A_i = \pm 1$

THEOREM: 3.6

If $A(\lambda) = A_0 + A_1\lambda + A_2\lambda^2 + ... + A_n\lambda^n$ be polynomial orthogonal of type I matrix. Here coefficient matrices $A_i{}'s$ are orthogonal of type I matrices.

The following statements are equivalent.

(i) $A(\lambda)$ is an polynomial orthogonal of type I matrix.

(ii) $A(\lambda)^{-1}$ is an polynomial orthogonal of type I matrix.

(iii) $A(\lambda)^T$ is an polynomial orthogonal of type I matrix.

(iv) $\overline{A(\lambda)}$ is an polynomial orthogonal of type I matrix.

(v) $A(\lambda)^*$ is an polynomial orthogonal of type I matrix.

Proof :

Let $A(\lambda) = A_0 + A_1\lambda + A_2\lambda^2 + ... + A_n\lambda^n$ be polynomial orthogonal of type I matrix. Here coefficient matrices $A_i{}'s$ are orthogonal of type I matrices. Where i = 1, 2, 3,......n.

To prove (i) ⇒ (ii).

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To prove $A(\lambda)^{-1}$ is a polynomial orthogonal of type I matrix. By the definition of polynomial orthogonal of type I matrix, all the coefficient matrices of $A(\lambda)$ are orthogonal of type I matrices. That is $A_0, A_1, A_2, ...A_n$ are orthogonal of type I matrices.

Suppose that $A_i^{'}s$ are orthogonal of type I matrices.

That is, $A_i^{\,k}(A_i^{\,T})^k = I_n$.

Taking inverse on both sides,

$$\left[A_i^{\,k}(A_i^{\,T})^k\right]^{-1} = [I_n]^{-1}$$

$$\left[A_i^{\,k}\right]^{-1}\left[(A_i^{\,T})^k\right]^{-1} = I_n$$

$$\left[A_i^{-1}\right]^k\left[(A_i^{-1})^T\right]^k = I_n \ \ where\ i = 1,2,3,........n$$

Hence A_i^{-1} are orthogonal of type I matrices.

Therefore $A_i{}'s$ are orthogonal of type I matrices. Hence all the coefficients of $A(\lambda)^{-1}$ are orthogonal of type I matrices. Therefore $A(\lambda)^{-1}$ is a polynomial orthogonal of type I matrix.

To prove (ii) ⇒ (iii).

Suppose $A(\lambda)^{-1}$ is a polynomial orthogonal of type I matrix to prove $A(\lambda)^T$ is a polynomial orthogonal of type I matrix. That is to prove all its coefficient matrices are orthogonal of type I matrices. Since $A(\lambda)^{-1}$ is a polynomial orthogonal of type I matrix.

This implies, $\left(A_i^{-1}\right)^k\left[\left(A_i^{-1}\right)^T\right]^k = I_n,\ i = 0, 1, 2, \cdots, n,$

Taking inverse on both sides,

$$\left[\left(A_i^{-1}\right)^k\left[\left(A_i^{-1}\right)^T\right]^k\right]^{-1} = [I_n]^{-1},$$

$$\left[\left[\left(A_i^{-1}\right)^T\right]^k\right]^{-1}\left[\left(A_i^{-1}\right)^k\right]^{-1} = I_n$$

$$\left[\left[\left(A_i^{-1}\right)^{-1}\right]^T\right]^k\left[\left(A_i^{-1}\right)^{-1}\right]^k = I_n$$

$$\left[\left(A_i\right)^T\right]^k\left(A_i\right)^k = I_n.$$

Taking transpose on both side

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$$\left[\left[(A_i)^T\right]^k (A_i)^k\right]^T = (I_n)^T$$

$$\left[A_i^T\right]^k \left[((A_i^T)^T)\right]^k = I_n$$

Hence $A(\lambda)^T$ is a polynomial orthogonal of type I matrix.

To prove (iii) ⇒ (iv)

Suppose $A(\lambda)^T$ is a polynomial orthogonal of type I matrix to prove $\overline{A(\lambda)}$ is a polynomial orthogonal of type I matrix. That is to prove all its coefficient matrices are orthogonal of type I matrices.

Since $A(\lambda)^T$ is a polynomial orthogonal of type I matrix.

This implies, $A_i^k \left(A_i^T\right)^k = I_n, i = 0, 1, 2, \cdots, n$,

Taking conjugate on both sides,

$$\Rightarrow \left[\overline{A_i^k \left(A_i^T\right)^k}\right] = \overline{I_n}$$

$$\Rightarrow \left(\overline{A_i}\right)^k \left(\left(\overline{A_i^T}\right)\right)^k = I_n, i = 0, 1, 2, \cdots, n$$

Therefore all $\overline{A_i}'s$ are orthogonal of type I matrices. Hence $\overline{A(\lambda)}$ is a polynomial orthogonal of type I matrix.

To prove (iv) ⇒ (v)

Suppose $\overline{A(\lambda)}$ is a polynomial orthogonal of type I matrix to prove $\overline{A(\lambda)}^T = A(\lambda)^*$ is a polynomial orthogonal of type I matrix. That is to prove all its coefficient matrices are orthogonal of type I matrices. That is $\left(A_i^*\right)^k \left((A_i^*)^T\right)^k = I_n, i = 0, 1, 2, \cdots, n$

Since $\overline{A(\lambda)}$ is a polynomial orthogonal of type I matrix.

This implies, $\left(\overline{A_i}\right)^k \left(\left(\overline{A_i}^T\right)^k\right) = I_n, i = 0, 1, 2, \cdots, n$

Taking transpose on both sides,

$$\left[\left(\overline{A_i}\right)^k \left(\left(\overline{A_i}^T\right)^k\right)^T\right]^T = (I_n)^T$$

$$\left(\left(\left(\overline{A_i}\right)^T\right)^k\right)^T \left(\left(\overline{A_i}\right)^k\right)^T = I_n$$

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Since $\left(\overline{A_i}\right)^T = A_i^*$

$$\left(\left(\left(\overline{A_i}\right)^T\right)^T\right)^k \left(\left(\overline{A_i}\right)^T\right)^k = I_n$$

$$\left((A_i^*)^T\right)^k \left(A_i^*\right)^k = I_n$$

Therefore all A_i^*'s are orthogonal of type I matrices. Hence $A(\lambda)^*$ is a polynomial orthogonal of type I matrix.

To prove (v) ⇒ (i)

Suppose $A(\lambda)^*$ is a polynomial orthogonal of type I matrix to prove $A(\lambda)$ is a polynomial orthogonal of type I matrix. That is to prove all its coefficient matrices are orthogonal of type I matrices.

Since A^* is a polynomial orthogonal of type I matrix.
This implies, $\left((A_i^*)^T\right)^k \left(A_i^*\right)^k = I_n$.
Taking * on both sides

$$\left(\left((A_i^*)^T\right)^k \left(A_i^*\right)^k\right)^* = \left(I_n\right)^*$$

$$\left(\left(\left(A_i^*\right)^T\right)^k\right)^* \left(\left(A_i^*\right)^k\right)^* = I_n$$

$$\left(\left(\left(A_i^*\right)^*\right)^T\right)^k \left(\left(A_i^*\right)^*\right)^k = I_n$$

$$A_i^k (A_i^T)^k = I_n$$

Therefore all A_i's are orthogonal of type I matrices. Hence $A(\lambda)$ is a polynomial orthogonal type I matrix.

4. CONCLUSION

Here we have extended some properties of orthogonal of type I matrices to polynomial orthogonal of type I matrices. All other properties can also be extended in a similar way.

REFERENCE

1. Abedal-Hamza Mahdi Hamza Orthogonal of Type I Matrices with Application Applied Mathematical Sciences, Vol. 11, 2017, no. 40, 1983 - 1994 HIKARI Ltd,
2. Buslowicz, M. and Kaczorek,T., "Reachability and Mini-mum Energy Control of Positive Linear Discrete-Time Systems with One Delay," *Proceedings of 12th Mediter-ranean Conference on Control and Automation, Kasadesi- Izmur, CD ROM,* 2004.

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 2, 2022, p. 1251 - 1258
https://publishoa.com
ISSN: 1309-345

3. Gohberg I., Lancaster P., and Rodman L., Invariant Subspaces of Matrices with Applications, Wiley, New York, 1986 and SIAM, Philadelphia, 2006.
4. Kaczorek, T., “Extension of the Cayley Hamilton Theorem for Continuous Time Systems with Delays,” *International Journal of Applied Mathematics and Computer Science,* Vol. 15, No. 2, (2005), pp. 231-234.
5. Kaczorek, T., “Vectors and Matrices in Automation and Electrotechnics,” Polish Scientific Publishers, Warsaw, 1988.
6. Mcrtizios, B.G. and Christodolous, M.A., “On the Generalized Cayley Hamilton Theorem,” *IEEE Transactions on Automatic Control,* Vol. 31, No. 1 (1986), 156-157.
7. Vardulakis A.I.G., “Linear Multivariable Control”, John Wiley, Chichester, UK, 1991.
8. Wolovich W.A., “Linear Multivariable Systems”, Springer Verlag, 1974.