JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 2, 2022, p. 1049 - 1057
https://publishoa.com
ISSN: 1309-3452

A Third Order Runge-Kutta Method Based on Linear Combination Of Arithmetic Mean, Harmonic Mean And Heronian Mean For Hybrid Fuzzy Differential Equation

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ABSTRACT
In this paper, the three methods namely the third order Runge-Kutta method based on Arithmetic mean, Harmonic mean and Heronian mean are embedded into one method which is called as the third order Runge-Kutta method based on linear combination of Arithmetic mean, Harmonic mean and Heronian mean is proposed to find the approximate solution of hybrid fuzzy differential equation. Here the numerical algorithm using the proposed method is given for solving hybrid fuzzy initial value problem. The effectiveness of the proposed method is tested through illustration. The solutions are given in tables and figures which shows that the proposed method is suitable for solving any real life problems which are modelled into Hybrid fuzzy differential equations.

Keywords: Hybrid fuzzy differential equations, Runge-Kutta method, Arithmetic mean, Harmonic mean, Heronian mean, Triangular fuzzy number.

1. Introduction

Control systems that are capable of controlling complex systems which have discrete time dynamics and continuous time dynamics can be modeled by hybrid systems. The differential systems containing fuzzy valued functions and interaction with a discrete time controller are named as hybrid fuzzy differential systems.

The Euler and Runge-Kutta methods were used by Pederson and Sambandham [10, 11] for solving hybrid fuzzy differential equations. In [10, 6, 7] the stability properties and comparision theorems were studied. Prakash and Kalai selvi [12] have studied the numerical solution of hybrid fuzzy differential equations by predictor-corrector three step method. The numerical solution of hybrid fuzzy differential equations by Runge-Kutta method of order five and the dependency problem in fuzzy computation was studied by Kanagarajan, Indrakumar and Muthukumar in [5]. The solution of hybrid fuzzy differential equations using Adams-Bashforth, Adams-Moulton and Predictor-Corrector methods are discussed in [3]. Narayanamoorthy et al in [9, 8] used Iterative procedure and Taylors series method to solve hybrid fuzzy differential equations. A numerical solution for hybrid fuzzy differential equations by Adams fifth order predictor-corrector method was studied by Jayakumar and Kanagarajan in [4]. Gethsi Sharmila have solved fourth order Runge-Kutta method based on geometric mean for hybrid fuzzy initial value problem and centroidal mean for fuzzy and hybrid fuzzy initial value problem in [1, 2]. Gethsi Sharmila and Evangelin Diana Rajakumari have solved Third order Runge-kutta method based on the linear combination of Arithmetic mean, Harmonic mean and Geometric mean in [13].

In this work, a third order Runge-Kutta Method based on linear combination of Arithmetic Mean, Harmonic Mean and Geometric Mean is proposed to solve Hybrid fuzzy differential equations.

2. Preliminaries

Definition:2.1 (Fuzzy Set)

If X is a collection of objects denoted generally by x, then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs $\tilde{A} = \left\{ (x, \mu_{\tilde{A}}(x)) / x \in X \right\}$, where $\mu_{\tilde{A}}(x)$ is called the membership function or grade of membership of x in $\tilde{A}$.

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Volume 13, No. 2, 2022, p. 1049 - 1057
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ISSN: 1309-3452

Definition: 2.2 (Membership Function)

The membership function values need not always be described by discrete values. Sometimes, these turn out to be as described by a continuous function.

The notation commonly used to denote the membership functions are:

(i) The membership function of a fuzzy set $\tilde{A}$ is denoted by $\mu_{\tilde{A}}(x)$, i.e.,

$\mu_{\tilde{A}}(x) : X \to [0,1]$.

(ii) The membership function of a fuzzy set $\tilde{A}$ has the following form

$\tilde{A} : X \to [0,1]$.

Types of Fuzzy Numbers :

The types of fuzzy numbers are based on the shapes of the fuzzy numbers. Commonly we use the triangular fuzzy number the most. Also we have ordinary fuzzy number (interval), trapezoidal fuzzy number, parallelogram fuzzy number and bell shaped fuzzy number.

Definition: 2.3 (Triangular fuzzy number)

It is a fuzzy number represented with three points as follows: $\tilde{A} = (a_1, a_2, a_3)$, this representation is interpreted as membership functions and holds the following condition

(i) a_1 to a_2 is increasing function

(ii) a_2 to a_3 is decreasing function

(iii) $a_1 \le a_2 \le a_3$

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & , x \prec a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & , a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{a_3 - a_2} & , a_2 \le x \le a_3 \\ 0 & , x \succ a_3 \end{cases}$$

Now, if you get crisp interval by α - cut operation, interval $\tilde{A}_\alpha$ shall be obtained as follow $\forall\ \alpha \in [0,1]$ from

$$\frac{a_1^{(\alpha)} - a_1}{a_2 - a_1} = \alpha \quad , \quad \frac{a_3 - a_3^{(\alpha)}}{a_3 - a_2} = \alpha$$

The Third order runge-kutta method based on a linear Combination of Arithmetic Mean (AM), Harmonic Mean (HaM) and Heronian Mean (HEM) for IVP was introduced by Khattri as follows

$$RM\ (k_1, k_2) = \frac{14\ AM\ (k_1, k_2) - HaM\ (k_1, k_2) + 32\ HEM\ (k_1, k_2)}{45}$$

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3. The Hybrid Fuzzy Differential Equations

The hybrid fuzzy differential equation

$$\dot{y}(t) = f\big(t, y(t), \lambda_k(y_k)\big), t \in [\,t_k,\ t_{k+1}\,] \qquad k = 0,1,2, \dots$$

$$y\,(\,t_0\,) \,=\, y_0 \qquad (3.1)$$

Where $t_k = 0$ is strictly increasing and unbounded, y_k denotes $y(t_k)$, $f : [t_0, \infty) \times R \times R \rightarrow R$ is continuous, and each $\lambda_k : R \rightarrow R$ is a continuous function. A solution to (3.1) will be a function $y : [t_0, \infty) \rightarrow R$ satisfying (3.1) . For k = 0, 1, 2, 3, … , let $f_k : [t_k, t_{k+1}) \times R \rightarrow R$ where, $f\ (t, y_k(t)) = f\ (t, y_k(t), \lambda_k(y_k))$.
The hybrid fuzzy differential equation in (3.1) can be written in expanded form as

$$\dot{y}(t) = \begin{cases} \dot{y}_0(t) = f\big(t, y_0(t), \lambda_0(y_0)\big) = f_0\big(t, y_0(t)\big), y_0(t_0) = y_0, t_0 \le t \le t_1 \\ \dot{y}_1(t) = f\big(t, y_1(t), \lambda_1(y_1)\big) = f_1\big(t, y_1(t)\big), y_1(t_1) = y_1, t_1 \le t \le t_2 \\ \vdots \\ \dot{y}_k(t) = f\big(t, y_k(t), \lambda_k(y_k)\big) = f_k\big(t, y_k(t)\big), y_k(t_k) = y_k, t_k \le t \le t_{k+1} \\ \vdots \end{cases} \qquad (3.2)$$

and a solution of (3.1) can be expressed as

$$y\,(t) \,=\, \begin{cases} y_0(t) = y_0, t_0 \le t \le t_1 \\ y_1(t) = y_1, t_1 \le t \le t_2 \\ \vdots \\ y_k(t) = y_k, t_k \le t \le t_{k+1} \\ \vdots \end{cases} \qquad (3.3)$$

We note that the solution of (3.1) is continuous and piecewise differentiable over $[t_0, \infty)$ and differentiable on each interval (t_k, t_{k+1}) for any fixed $y_k \in R$ and k = 0, 1, 2, … .

4. A Third order Runge-Kutta method based on linear combination of Arithmetic mean, Harmonic mean and Heronian mean for hybrid fuzzy differential equation

Consider the IVP (3.1) with crisp initial condition $y\ (t_0) = y_0 \in R$ and $t \in [t_0, T]$. Let the exact solution $[\,Y(t)\,]_r = \big[\,\underline{Y}(t;r), \overline{Y}(t;r)\big]$ is approximated by some $[\ y\ (t)\,]_r = \big[\,\underline{y}(t;r), \overline{y}(t;r)\big]$ from the equation $y_{n+1} = y_n + \frac{h}{2}\left[\sum_{i=1}^{2} Means\right]$
where means includes Arithmetic Mean(AM), Geometric Mean (GM), Contraharmonic Mean (CoM), Centroidal Mean (CeM), Root Mean Square (RM), Harmonic Mean(HaM), and Heronian Mean (HeM) which involves $k_{i,}\ 1 \le i \le 4$, where,

$$k_1 = f(t_n,\ y_n)$$

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https://publishoa.com
ISSN: 1309-3452

$$k_2 = f(t_n + a_1\ h,\ y_n + a_1\ h\ k_1)$$

$$k_3 = f(t_n + (a_2 + a_3)\ h,\ y_n + a_2\ h\ k_1 + a_3\ h\ k_2)$$

where the parameters for the Combination of Arithmetic mean, Harmonic mean and Heronian mean are

$$a_1 = \frac{2}{3},\ a_2 = \frac{-148}{405},\ a_3 = \frac{418}{405}$$

The third order formulae based on Runge- Kutta scheme using the Combination of Arithmetic mean, Harmonic mean and Heronian mean are

$$y_{n+1} = y_n + \frac{h}{90}\left\{7\,(k_1 + 2k_2 + k_3) - \left(\frac{2k_1k_2}{k_1+k_2} + \frac{2k_2k_3}{k_2+k_3}\right) + \frac{32}{3}\left(k_1 + 2k_2 + k_3 + \sqrt{k_1k_2} + \sqrt{k_2k_3}\right)\right\}$$

with the grid points $a = t_0 \le t_1 \le \ldots \le t_N = b$ and $h = \frac{(b-a)}{N} = t_{i+1} - t_i$

We define

$$\underline{y}_{k,n+1}(r) - \underline{y}_{k,n}(r) = \sum_{i=1}^{3} w_i \underline{k}_i(t_{k,n}; y_{k,n}(r))$$

$$\overline{y}_{k,n+1}(r) - \overline{y}_{k,n}(r) = \sum_{i=1}^{3} w_i \overline{k}_i(t_{k,n}; y_{k,n}(r))$$

where w_1, w_2, w_3 are constants,

$$\underline{k}_1(t_{k,n}; y_{k,n}(r)) = min\left\{f(t_{k,n}, u, \lambda_k(u_k))\backslash u \in \left[\underline{y}_{k,n}(r), \overline{y}_{k,n}(r)\right],\ \ u_k \in \left[\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)\right]\right\}$$

$$\overline{k}_1\left(t_{k,n}; y_{k,n}(r)\right) = max\left\{f\left(t_{k,n}, u, \lambda_k(u_k)\right)\backslash u \in \left[\underline{y}_{k,n}(r), \overline{y}_{k,n}(r)\right], u_k \in \left[\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)\right]\right\}$$

$$\underline{k}_2(t_{k,n}; y_{k,n}(r)) = min\left\{\begin{matrix} f(t_{k,n} + \frac{2}{3}h_k, u, \lambda_k(u_k))\backslash\ u \in [\underline{z}_{k_1}(t_{k,n}; y_{k,n}(r)), \overline{z}_{k_1}(t_{k,n}; y_{k,n}(r))], \\ u_k \in \left[\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)\right] \end{matrix}\right\}$$

$$\overline{k}_2(t_{k,n}; y_{k,n}(r)) = max\left\{\begin{matrix} f(t_{k,n} + \frac{2}{3}h_k, u, \lambda_k(u_k))\backslash\ u \in [\underline{z}_{k_1}(t_{k,n}; y_{k,n}(r)), \overline{z}_{k_1}(t_{k,n}; y_{k,n}(r))], \\ u_k \in \left[\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)\right] \end{matrix}\right\}$$

$$\underline{k}_3(t_{k,n}; y_{k,n}(r)) = min\left\{\begin{matrix} f(t_{k,n} + \frac{2}{3}h_k, u, \lambda_k(u_k))\backslash\ u \in [\underline{z}_{k_2}(t_{k,n}; y_{k,n}(r)), \overline{z}_{k_2}(t_{k,n}; y_{k,n}(r))], \\ u_k \in \left[\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)\right] \end{matrix}\right\}$$

$$\overline{k}_3(t_{k,n}; y_{k,n}(r)) = max\left\{\begin{matrix} f(t_{k,n} + \frac{2}{3}h_k, u, \lambda_k(u_k))\backslash\ u \in [\underline{z}_{k_2}(t_{k,n}; y_{k,n}(r)), \overline{z}_{k_2}(t_{k,n}; y_{k,n}(r))], \\ u_k \in \left[\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)\right] \end{matrix}\right\}$$

where

$$\underline{z}_{k_1}(t_{k,n}; y_{k,n}(r)) = \underline{y}_{k,n}(r) + \frac{2}{3}h_k\underline{k}_1(t_{k,n}; y_{k,n}(r))$$

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https://publishoa.com
ISSN: 1309-3452

$$\overline{z}_{k_1}\left(t_{k,n}; y_{k,n}(r)\right) = \overline{y}_{k,n}(r) + \frac{2}{3} h_k \overline{k}_1\left(t_{k,n}; y_{k,n}(r)\right)$$

$$\underline{z}_{k_2}\left(t_{k,n}; y_{k,n}(r)\right) = \underline{y}_{k,n}(r) - \frac{148}{405} h_k \underline{k}_1\left(t_{k,n}; y_{k,n}(r)\right) + \frac{418}{405} h_k \underline{k}_2\left(t_{k,n}; y_{k,n}(r)\right)$$

$$\overline{z}_{k_2}(t_{k,n}; y_{k,n}(r)) = \overline{y}_{k,n}(r) - \frac{148}{405} h_k \overline{k}_1(t_{k,n}; y_{k,n}(r)) + \frac{418}{405} h_k \overline{k}_2(t_{k,n}; y_{k,n}(r))$$

define

$$S_k\left[t_{k,n}; \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)\right] = \left\{\begin{array}{l} 7\left[\underline{k}_1(t_{k,n}, y_{k,n}(r)) + 2\underline{k}_2(t_{k,n}, y_{k,n}(r)) + \underline{k}_3(t_{k,n}, y_{k,n}(r))\right] - \\ \left[\begin{array}{l} \dfrac{2\underline{k}_1(t_{k,n}, y_{k,n}(r))\underline{k}_2(t_{k,n}, y_{k,n}(r))}{\underline{k}_1(t_{k,n}, y_{k,n}(r)) + \underline{k}_2(t_{k,n}, y_{k,n}(r))} + \\ \dfrac{2\underline{k}_2(t_{k,n}, y_{k,n}(r))\underline{k}_3(t_{k,n}, y_{k,n}(r))}{\underline{k}_2(t_{k,n}, y_{k,n}(r)) + \underline{k}_3(t_{k,n}, y_{k,n}(r))} \end{array}\right] + \\ \dfrac{32}{3}\left[\dfrac{\underline{k}_1(t_{k,n}, y_{k,n}(r)) + 2\underline{k}_2(t_{k,n}, y_{k,n}(r)) + \underline{k}_3(t_{k,n}, y_{k,n}(r)) +}{\sqrt{\underline{k}_1(t_{k,n}, y_{k,n}(r))\underline{k}_2(t_{k,n}, y_{k,n}(r))} + \sqrt{\underline{k}_2(t_{k,n}, y_{k,n}(r))\underline{k}_3(t_{k,n}, y_{k,n}(r))}}\right] \end{array}\right\}$$

$$T_k\left[t_{k,n}; \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)\right] = \left\{\begin{array}{l} 7\left[\overline{k}_1(t_{k,n}, y_{k,n}(r)) + 2\overline{k}_2(t_{k,n}, y_{k,n}(r)) + \overline{k}_3(t_{k,n}, y_{k,n}(r))\right] - \\ \left[\begin{array}{l} \dfrac{2\overline{k}_1(t_{k,n}, y_{k,n}(r))\overline{k}_2(t_{k,n}, y_{k,n}(r))}{\overline{k}_1(t_{k,n}, y_{k,n}(r)) + \overline{k}_2(t_{k,n}, y_{k,n}(r))} + \\ \dfrac{2\overline{k}_2(t_{k,n}, y_{k,n}(r))\overline{k}_3(t_{k,n}, y_{k,n}(r))}{\overline{k}_2(t_{k,n}, y_{k,n}(r)) + \overline{k}_3(t_{k,n}, y_{k,n}(r))} \end{array}\right] + \\ \dfrac{32}{3}\left[\dfrac{\overline{k}_1(t_{k,n}, y_{k,n}(r)) + 2\overline{k}_2(t_{k,n}, y_{k,n}(r)) + \overline{k}_3(t_{k,n}, y_{k,n}(r)) +}{\sqrt{\overline{k}_1(t_{k,n}, y_{k,n}(r))\overline{k}_2(t_{k,n}, y_{k,n}(r))} + \sqrt{\overline{k}_2(t_{k,n}, y_{k,n}(r))\overline{k}_3(t_{k,n}, y_{k,n}(r))}}\right] \end{array}\right\}$$

The exact solutions at $t_{k,n+1}$ is given by

$$\underline{Y}_{k,n+1}(r) \approx \underline{Y}_{k,n}(r) + \frac{h_k}{90} S_k\left[t_{k,n}; \underline{Y}_{k,n}(r), \overline{Y}_{k,n}(r)\right]$$
$$\overline{Y}_{k,n+1}(r) \approx \overline{Y}_{k,n}(r) + \frac{h_k}{90} T_k\left[t_{k,n}; \underline{Y}_{k,n}(r), \overline{Y}_{k,n}(r)\right]$$

The approximate solutions at $t_{k,n+1}$ is given by

$$\underline{y}_{k,n+1}(r) \approx \underline{y}_{k,n}(r) + \frac{h_k}{90} S_k\left[t_{k,n}; \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)\right]$$

$$\overline{y}_{k,n+1}(r) \approx \overline{y}_{k,n}(r) + \frac{h_k}{90} T_k\left[t_{k,n}; \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)\right]$$

5. Numerical Examples

Before illustrating the numerical solution of Hybrid Fuzzy Initial Value Problem consider the fuzzy Initial Value Problem

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 2, 2022, p. 1049 - 1057
https://publishoa.com
ISSN: 1309-3452

$$\begin{cases} y'(t) = y(t),\ t \in [0,1] \\ y(0;r) = [0.75+0.25r, 1.125-0.125r],\ 0 \le \mathrm{r} \le 1 \end{cases}$$

The exact solution is given by

$$Y(t;r) = [(0.75+0.25r)e^t, (1.125-0.125r)e^t],\ 0 \le \mathrm{r} \le 1\ .$$

at t=1 we get

$$Y(1;r) = [(0.75+0.25r)e^1, (1.125-0.125r)e^1],\ 0 \le \mathrm{r} \le 1$$

By the third order Runge –Kutta method based on the combination of Arithmetic mean, Harmonic mean and Heronian Mean with N=2 , gives

$$y(1.0;r) = [(0.75+0.25r)(c_{0,1})^2, (1.125-0.125r)(c_{0,1})^2],\ 0 \le \mathrm{r} \le 1$$

where

$$c_{0,1} = 1 + \frac{h}{90}\left\{7\left[4 + \frac{162}{81}h + \frac{836}{1215}h^2\right] - \left[\frac{2+\frac{4}{3}h}{2+\frac{2}{3}h} + \frac{2+\frac{1080}{405}h + \frac{2752}{1215}h^2 + \frac{3344}{3645}h^3}{2+\frac{540}{405}h+\frac{836}{1215}h^2}\right]\right.$$
$$\left. + \frac{32}{3}\left[4 + \frac{810}{405}h + \frac{836}{1215}h^2 + \sqrt{1+\frac{2}{3}h} + \sqrt{1+\frac{540}{405}h+\frac{1376}{1215}h^2+\frac{1672}{3645}h^3}\right]\right\}$$

Now consider the following Hybrid fuzzy initial value problem

$$\begin{cases} y'(t) = y(t) + m\ (t)\ \lambda_k(y(t_k)),\ t \in [t_k, t_{k+1}]\ ,\ t_k = k,\ k = 0,1,2,\dots \\ y(t;r) = \ [\ (0.75+0.25r)e^t,\ (1.125-0.125r)e^t\ ],\quad 0 \le r \le 1 \end{cases} \tag{5.1}$$

where

$$m(t) = \begin{cases} 2\ (t\ (\bmod 1\ ))\quad ,\ if\ \ t\ (\bmod 1\ ) \le 0.5, \\ 2\ (1 - t\ (\bmod 1\ )\ )\ ,\ if\ \ t\ (\bmod 1\ ) \ge 0.5 \end{cases}$$

$$\lambda_k\ (\ y\ (\ t_k\ )\ ) = \begin{cases} 0 \qquad ,\ if\ \ k\ = 0, \\ \mu\ ,\ if\ \ k\ \in \{\ 1,2,\dots\} \end{cases}$$

The hybrid fuzzy Initial value problems is equivalent to the following system of fuzzy Initial value problems.
The exact solution for $t \in [0,2]$ is given by

$$\begin{cases} y_0\,'(t) = y_0(t),\ t \in [0,1] \\ y_0(0;r) = [(0.75+0.25r)e, (1.125-0.125r)e],\ 0 \le \mathrm{r} \le 1 \\ y_i^{'}(t) = y_i(t) + m(t)y_{i-1}(t),\ \mathrm{t} \in \left[t_i, t_{i+1}\right], y_i(t) = y_{i-1}(t_i), i = 1, 2, \dots, \end{cases}$$

$y(t) + m(t)\lambda_k(y(t_k))$ is continuous function of t, x and $\lambda_k(y(t_k))$, for each k=0,1,2,…, the fuzzy Initial value problem

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Volume 13, No. 2, 2022, p. 1049 - 1057
https://publishoa.com
ISSN: 1309-3452

$$\begin{cases} y'(t) = y(t) + m(t)\lambda_k(y(t_k)),\ t \in [t_k, t_{k+1}], t_k = k \\ y(t_k) = y_{t_k}, \end{cases}$$

has a unique solution on $[t_k, t_{k+1}]$. To numerically solve the hybrid fuzzy IVP we will apply the modified third order Runge – Kutta method based on Arithmetic Mean, Harmonic Mean and Heronian Mean for hybrid fuzzy differential equation with N=2 to obtain $y_{1,2}(r)$ approximating x(2.0; r) .

Let $f : [0,\infty) \times R \times R \to R$ be given by

$$f(t, y, \lambda_k(y(t_k))) = y(t) + m(t)\lambda_k(y(t_k)), t_k = k, k = 0,1,2,...,$$

Where $\lambda_k : R \to R$ is given by

$$\lambda_k(y) = \begin{cases} 0, \text{ if k=0} \\ y, \text{ if k} \in \{1,2,...,\} \end{cases}$$

Since the exact solution of for t϶[1, 1.5] is $Y(t;r) = Y(1;r)(3e^{t-1} - 2t), 0 \le r \le 1,\ \text{Y(1.5;r)=Y(1;r)}(3\sqrt{e}\text{-3}), 0 \le \text{r} \le 1.$

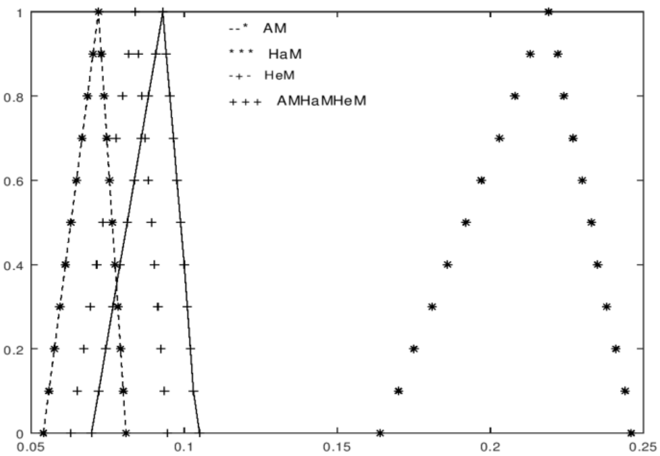
Then Y(1.5;r) is approximately 5.29 and $y_{1,1}$ is approximately 5.243. Since the exact solution for t[1.5, 2] is $Y(t;r) = Y(1;r)(2t - 2 + e^{t-1.5}(3\sqrt{e} - 4)), 0 \le r \le 1$.

Therefore $\text{Y(2.0;r)=Y(1;r)(2+3e-4}\sqrt{e}).$

The approximate solution, exact solution and absolute error using the third order Runge-Kutta method based on a linear combination of Arithmetic Mean, Harmonic Mean and Heronian Mean for the r-level set with h = 0.5 and t = 2 of the example is Given below.

r	**t**	**approximate solution**		**exact solution**		**Error**	
0	2	7.194701	10.792051	7.257732	10.886598	6.30E-02	9.45E-02
0.1	2	7.434524	10.672139	7.499656	10.765635	6.51E-02	9.35E-02
0.2	2	7.674347	10.552228	7.741581	10.644673	6.72E-02	9.24E-02
0.3	2	7.914171	10.432316	7.983505	10.523711	6.93E-02	9.14E-02
0.4	2	8.153994	10.312404	8.225429	10.402749	7.14E-02	9.03E-02
0.5	2	8.393817	10.192493	8.467354	10.281787	7.35E-02	8.93E-02
0.6	2	8.633641	10.072581	8.709278	10.160824	7.56E-02	8.82E-02
0.7	2	8.873464	9.952669	8.951202	10.039862	7.77E-02	8.72E-02
0.8	2	9.113287	9.832758	9.193127	9.9189	7.98E-02	8.61E-02
0.9	2	9.353111	9.712846	9.435051	9.797938	8.19E-02	8.51E-02
1	2	9.592934	9.592934	9.676976	9.676976	8.40E-02	8.40E-02

The Graphical representation of the exact and approximate solution of the third order Runge-Kutta method based on the linear combination of Arithmetic Mean, Harmonic Mean and Heronian Mean for the above example . where t = 2, h = 0.5 is given below



6. Conclusion

In this paper, the first order hybrid fuzzy differential equation is solved using the third order Runge-Kutta method based on the linear combination of Arithmetic Mean, Harmonic Mean and Heronian Mean. The solution of the proposed method is compared with the Arithmetic Mean, Harmonic Mean and Heronian Mean methods. From the numerical example, we can conclude that the proposed method is suitable for solving hybrid fuzzy initial value problem.

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