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Three Aspects for Special Cases of Linear Programming Problems

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ABSTRACT

This study is devoted to discuss the special cases that occur in linear programming problems (LPP). We have discussed each of them with three aspects. The aspect represents the algebraic base that has been shown with help of POM-QM software, and TORA software, the second one focuses on the geometric representations of three-dimensional space, the third aspect deals with the economic and administrative meaning. These ideas have been shown in geometrical meanings of three dimensional space by using the MAPL software.

Introduction:

Problems in linear programming can be solved by different methods.The most famous one is the simplex method. Special cases of the solutions within such method can be observed with indices in the solution tables or the final table in the simplex.

In addition,linear programming problemsthat can be solved graphically it provides an efficient tool that clarifying the elements of solution and the relationships between the variables of the problem.This method has also some special cases when representing the problems.Many researchers presented these special cases,either by simplex method alone or by graphical method in 2- dimensions, [1-3].

Literature Review:

In 2014, Harvey mentioneda method for solving linear programming models.The methodis called requirements space representation and use of three-dimensions representation.The constraints in this method are represented in the form of vectors, and slack variables are added to those vectors, which are located on the same coordinate.The constraint is represented by a plane composed of these vectors. In particular, the special cases has only represented at the two-dimensions case, [1].

In 1976, Hillier Lieberman solves linear programming problems in the 3-dimensions by representing the decision variables with three coordinates and representing the problem constraints in the form of an equation while ignoring the additive variables.The feasible solution region is polyhedron rather than the polygon it checks all surfaces representing the constraints and does not present special cases in this way, [2].

Mishra, S.K., & Ram, B. (2017).They presented linear programming problems in the form of matrices. Use the graph to represent the problems that he dealt with .Use Matlab to display the graphical method in two dimensions . And display the case of the unbounded solution without other special cases[3]

R. Clark Robinson (2013).displayed in the graphical solution method in two coordinate and one special case. It was shown in the algebraic-simplex solution method,[4].

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Winston presenting special cases,practical examples, shortening theirrepresentation only in 2-dim and displayingthemethods of representation in 3-dim. The constraint is represented by three points,and each of them is represented by a plane and his half-spaces. The representation isin a way that achieves that constraint, while the other side is not achieved,and the solution is one of the corner points inequality, [5].

Catherine, plotted LPP system with two dimensions for solving and illustrate the feasible region .This representation was used to solve a practical problem. The theoretical aspect was presented by the set of a polyhedron and the area obtained from the intersection of plans, edges, extreme pointswhich represented the problem constraints.[6]

In this research, we present three aspects for solving these problems in threedimensional space. **The first aspect** of the solution represents the algebraic form which is presented through the simplexmethodthrough the POM-QM software aseach case shows indications that represent the special case of the solution. **The secondaspect** represents the solution in a geometricalway.This aspect involves solutions of three dimensional space that has been presented with help of MAPL software.**Thethird aspect** is the economic or administrativemeanings that appears in the representation of these cases.

The special cases that appear in solving linear Programming problems(LPP) are:

1. Multiple optimal solution case.
2. Degeneracy case and Redundant constraint case.
3. Unbound optimal solution case.
4. Infeasible solution casa.
5. Unique solution.

1. Multiple Optimal Solution Case:

The first aspect:the algebraic aspect is the case that appears in the algebraic solution by the simplex method in the last table (the optimal solution) when we want to choose the most negative element.We find that there are more than one pivot column having the same value (the pivot columns of the basic variables or in some cases the non-basic variables having an equal value).By taking any pivot columns,we get the same objective function**.**

Production departments	Types of chemical fluids			Available hours for each department
	Glucose	Saline	Ringer	
Section 1	2	6	2	24
Section2	2	3	6	36
Profit/unit	1	3	1	

Example 1.1A chemical company produces three types of chemical fluids, each of which requires pass through two production departments, respectivelyin order to include the necessary time in each production department and profit of the product as in the table below. The possible quantity of the production can be calculated in every fluid to the maximize profit.

$$Maximize\ Z = x_1 + 3x_2 + x_3$$

$$subject\ to \quad 2x_1 + 6x_2 + 2x_3 \leq 24, 2x_1 + 3x_2 + 6x_3 \leq 36, \quad x_1, x_2, x_3 \geq 0$$

Solution by the simplex method is shown in Table 1.1

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In the first table, the most negative element was chosen, which is -3 in the second table,we note that there are more than one pivot column with the same value,the basic variables (x_1, x_2, x_3)=0, as well as the non-basic variablex_5=0.

Iteration1						
Basic	x_1	x_2	x_3	S_{x_4}	S_{x_5}	Solution
$Max(Z)$	-1	-3	-1	0	0	0
S_{x_4}	2	6	2	1	0	24
S_{x_5}	2	3	6	0	1	36
Iteration2						
Basic	x_1	x_2	x_3	Sx_4	Sx_5	Solution
$Max(Z$	0	0	0	0.5	0	12
x_2	0.3	1	0.3	0.17	0	4
S_{x_5}	1	0	5	-0.5	1	24
Iteration3						
Basic	x_1	x_2	x_3	S_{x_4}	S_{x_5}	Solution
$Max(Z)$	0	0	0	0.5	0	12
x_2	0.27	1	0	0.2	-0.07	2.40
x_3	0.20	0	1	-0.1	0.2	4.80

Table1.1:Iterations of multiple solution

When,we take any solution,we find out that the Z value remains equal to 12,
$X^* =$
$(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*)$=(12,0,0,0,12), $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*)$=(0,2,4,0,0),and$(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*)$ =(0,4,0,0,24).

In fact, we can find any interior point between the two extreme points(a, b) that represents the multiple solution,if the optimal multiple solution is between the two extremes points(a, b).

The second aspect:the geometric aspect is the case (multiple optimal solutions) appears when one of the lines representing the constraints in the case of two dimensions which is parallel or identical to the line of the objective function. It is clear that this is true because the lines have the same slope.For three dimensional space, we deals with planes instead of lines, the feasible region for the solution is represented by the intersection of the constraints planes. So, whenthe plane of one of the constraints is parallel to the plane of the objective function, thenthe solution is represented by a straight line which is obtained by the intersection. This means there is more than one optimal solution to this problem, and they all yield the same value to the objective function.The multiple optimal solution can be clarified by testing the extreme points of the feasible region solution.If the optimal value is located on more than one extreme point,then this problem has more than one optimal solution.

Con 1:(12,0,0),(0,4,0) and (0,0,12), Con2: (18,0,0),(0,12,0) and (0,0,6)

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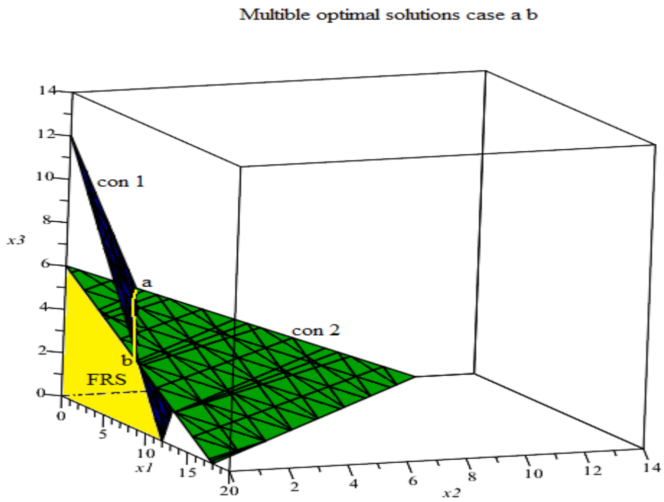

Figure1.1:feasible region multiple solution

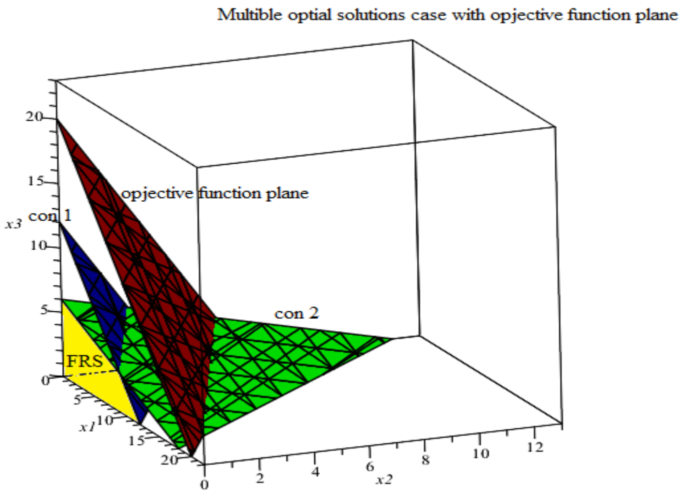

Figure1-2: *objective functionplane with constraints*

The Third Aspect: Economic and administrative,we note that studying the results in the previous table,the following are concluded: -

- The multiple solutions that have different values in each case, the amount of the profit remains settle (12$) even we change those values. The solution of x_2 produce 4 units, and the solution of x_2, x_3 produce 2.4, 4.8 units, respectively, yield the same profit.
- The secondtype of Salinefluid and the third type of Ringerfluid appeared in the optimal solution, where it refers to the quantity that must be produced and sold of the product x_2,x_3 is by 2 L/h,4 L/h, respectively.
- It was found from the optimal solution that if the company committed to implementing the plan decided by the optimal solution,it would achieve profits of 12$/h.
- We note that the contribution of the first product which is Glucose to the value of the profit is zero. In the case, the company does not want to produce the first fluid(Glucose), so the second and third fluids are sufficient,they achieve the same profit.

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2.Degeneracy case:

The first aspect**:**(Algebraic aspect or Simplex) When applying the simplex method and note the feasible condition in the solution, it appears as a special case. When choosing the departing variable, we find that the smaller ratio represents two equal ratios. If one of them is chosen optional or the one which is more important in the objective function. the subsequent table in which the other variable has a value zero and this does not affect the value of the objective function (it remains constant). Whereas the new row is calculated in the new row through the following law:

$a'_{ij=}a_{ij} - (\frac{a_{ik\times} a_{rj}}{a_{rk}})$;∀i≠ $r, j \neq k,$ i=1,2,…,n j=1,2,…,m

Example 2.1 An Iraqi company produces 3 sizes of carpets.The carpets manufacturing has three phases, the cutting, folding, and packaging. For the purpose of selling them in markets,as the time available for cutting operations is at least 3 hours per day, and the time available for folding operations is at most 12 hours per day.The time available for packaging process is at most 3 hours a day, and the rest of the details are shown in the table2.1.

Production departments	Product type of carpets			Available time
	A	B	C	
Cut	1	1	1	3
Folding	2	1	4	12
Packaging	0	0	1	3
Gain alone height	3	2	7	

$$Maximize = 3x_1 + 2x_2 + 7\,x_3$$

$subjet\ to: x_1 + x_2 + x_3 \geq 3$, $2x_1 + x_2 + 4x_3 \leq 12$,

$$x_3 \leq 3 \quad , \qquad x_1, x_2, x_3 \geq 0$$

By using QM-POM programming

Cj	Basic Variables	3 X1	2 X2	7 X3	0 artfcl 1	0 surplus 1	0 slack 2	0 slack 3	Quantity
0	slack 3	0	0	1	0	0	0	1	3
Iteration 4									
	cj-zj	-6.7	-5.0	0	-7.0	7.0	0	0	
7	X3	1	1	1	1	-1	0	0	3
0	slack 2	-2	-3	0	-4	4	1	0	0
0	slack 3	-1	-1	0	-1	1	0	1	0
Iteration 5									
	cj-zj	-3.2	0.25	0	0	0	-1.75	0	
7	X3	0.5	0.25	1	0	0	0.25	0	3
0	surplus 1	-0.5	-0.75	0	-1	1	0.25	0	0
0	slack 3	-0.5	-0.25	0	0	0	-0.25	1	0
Iteration 6									
	cj-zj	-3.7	0	-1.0	0	0	-2.0	0	
2	X2	2	1	4	0	0	1	0	12
0	surplus 1	1	0	3	-1	1	1	0	9
0	slack 3	0	0	1	0	0	0	1	3

Table 2-1*: Iterations of degeneracy solution*

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When we cancel the third constraint the optimal solution, it will remain at the same solution.

As shown below.table (2-2)

Cj	Basic Variables	.3 X1	2 X2	7 X3	0 artfcl 1	0 surplus 1	0 slack 2	Quantity
Iteration 3								
	cj-zj	0	1.7	6.7	-0.3	0.3	0	
.3	X1	1	1	1	1	-1	0	3
0	slack 2	0	-1	2	-2	2	1	6
Iteration 4								
	cj-zj	-6.7	-5.0	0	-7.0	7.0	0	
7	X3	1	1	1	1	-1	0	3
0	slack 2	-2	-3	0	-4	4	1	0
Iteration 5								
	cj-zj	-3.2	0.25	0	0	0	-1.75	
7	X3	0.5	0.25	1	0	0	0.25	3
0	surplus 1	-0.5	-0.75	0	-1	1	0.25	0
Iteration 6								
	cj-zj	-3.7	0	-1.0	0	0	-2.0	
2	X2	2	1	4	0	0	1	12
0	surplus 1	1	0	3	-1	1	1	9

Table 2-2: *Iterations of degeneracy with the same objective value same*

Thus, the third constraint isredundancy and the solution is degenerate.

The Second Aspect:the geometric aspectof this example, the solution represent the intersection point of the constraints planes. When we omitted one of them and does not affect the feasible region solution, then that plane represents the redundant constraint.

By using **Maple** software, the geometric interpretation above can be shown in the figure 3.1

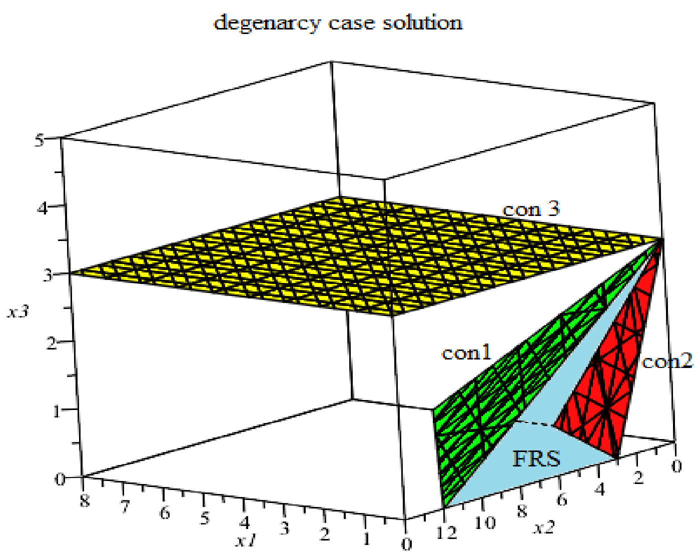

Figure 2-1*: feasible region of degeneracy solution*

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The Third Aspect:(economic and administrative) the optimal solution tells us that the carpet of the third sizes,when sold or not sold, does not affect the profit return.

3- Unbounded Case:

The algebraic aspect: This special case (unbounded) appears in the simplex method, which means that there are many solutions. This happens in the issues of maximizing profits when there is the possibility of increasing profits to infinity without violating the constraints of the problem. If wecome across this situation in the issues of practical life, this indicates the LPP has been formulated in an inappropriate way as it is impossible to increase profits practically in the form unbounded. This problem can be discovered when using the simplex method before reaching the final solution when we want to determine which factors should be taken out of the solution by finding the ratio through the law:$p = \left\{ \text{smallest ratio} \frac{y_{io}}{y_{iq}} : y_{iq} > 0 \right\}$ The result is neither negative, infinity, nor a combination of both.

Example3.1:A company for the manufacture of furniture was able to produce (offices, chairs, and doors),where these products pass through three industrial departments (carpentry,assembly and dyeing),where the offices need to spend (1,4,2) hours, respectively,and a profit from this product equals 6$. The second product(chairs)needs to spend (2,2)hours to do the carpentry and assembly process, respectively,where the profit from this product is 5$. The last product,which is the doors,needs 3 hours for each section, andthe profit is 3$. knowing the time available is at least (18,25,18) hours per day for each industrial department.

$$Maximize\ \ Z = 6x_1 + 5x_2 + 3x_3$$

Subjectto: $x_1 + \ 2\,x_2 + 3x_3 \geq 18, x_1 + 2x_3 + 3x3 \geq 25$ xx $2x_1 \ + 3x3 \geq 18,$ $x_1, x_2, x_3 \geq 0$

Cj	Basic Variables	6 X1	5 X2	3 x3	0 artfcl 1	0 surplus 1	0 artfcl 2	0 surplus 2	0 artfcl 3	0 surplus 3	Quantity
Iteration 1											
	cj-zj	7	4	9	0	-1	0	-1	0	-1	
0	artfcl 1	1	2	3	1	-1	0	0	0	0	18
0	artfcl 2	4	2	3	0	0	1	-1	0	0	25
0	artfcl 3	2	0	3	0	0	0	0	1	-1	18
Iteration 2											
	cj-zj	4	-2.0	0	-3	2.0	0	-1	0	-1	
3	x3	0.3333	0.6667	1	0.3333	-0.3333	0	0	0	0	6
0	artfcl 2	3	0	0	-1	1	1	-1	0	0	7
0	artfcl 3	1	-2	0	-1	1	0	0	1	-1	0
Iteration 3											
	cj-zj	0	6	0	1	-2	0	-1	-4	3	
3	x3	0	1.3333	1	0.6667	-0.6667	0	0	-0.3333	0.3333	6
0	artfcl 2	0	6	0	2	-2	1	-1	-3	3	7
6	X1	1	-2	0	-1	1	0	0	1	-1	0

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Iteration 4											
	cj-zj	0	0	0	-1.0	0	-1	0	-1	0	
3	x3	0	0	1	0.2222	-0.2222	-0.2222	0.2222	0.3333	-0.3333	4.4444
5	X2	0	1	0	0.3333	-0.3333	0.1667	-0.1667	-0.5	0.5	1.1667
6	X1	1	0	0	-0.3333	0.3333	0.3333	-0.3333	0	0	2.3333
Iteration 5											
	cj-zj	0	0	0	-0.3333	0.3333	-2.1667	2.1667	1.5	-1.5	
3	x3	0	0	1	0.2222	-0.2222	-0.2222	0.2222	0.3333	-0.3333	4.4444
5	X2	0	1	0	0.3333	-0.3333	0.1667	-0.1667	-0.5	0.5	1.1667
6	X1	1	0	0	-0.3333	0.3333	0.3333	-0.3333	0	0	2.3333
Iteration 6											
	cj-zj	0	0	-9.75	-2.5	2.5	0	0	-1.75	1.75	
0	surplus 2	0	0	4.5	1.0	-1.0	-1	1	1.5	-1.5	20.0
5	X2	0	1	0.75	0.5	-0.5	0	0	-0.25	0.25	4.5
6	X1	1	0	1.5	0	0	0	0	0.5	-0.5	9.0

Table 3-1:*Iterations of Unbounded case*

The second aspect:where this special case (unbounded) shows us in the graphical (geometric) method that the area of the solution resulting from the intersection of the planes which represented by the constraints and appears to us as an open area without an end.

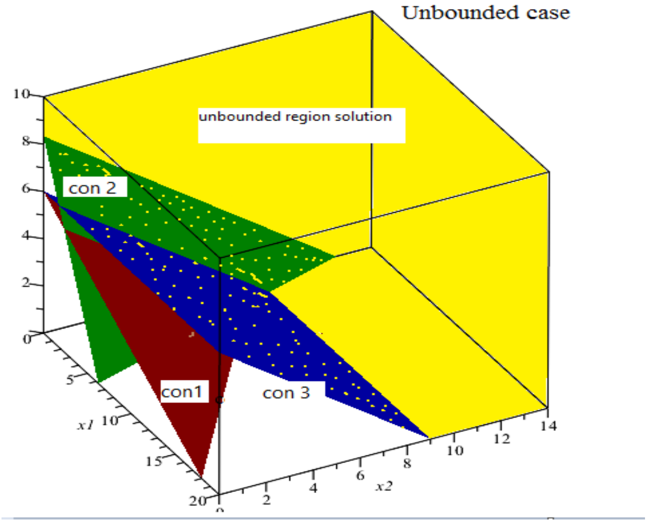

Figure 3-1:Unbounded feasible solution

The third aspect: Economic and management

where this case shows us a theoretical case only and far from practical reality asit is not possible to increase one or more variables in the problem and consequently profit without violating any of the restrictions of the problem. For instance, the number of employees, machines, or time, all of these resources (restrictions) are limited and cannot be infinite.

4- Infeasible Solution Case:

The algebraic aspect: This case appears when applying the simplex method, we notice that when we reach the last table (the optimal solution), we find that all the numbers in the last row in the case of maximization have a positive value (≥ 0). However, there is still an artificial variable that is still occurringin the solution in the basic variable and with a positive value in the case of maximization.

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Example4.1:One of the shops sells three types of clothes (socks, shirts, and veil). The profit of the socks is \$0.5, the profit of the shirt is \$5, and the profit of the veil is \$3. The socks need 1 minute for each of the salesmantime and the time of the cashier.As for the shirt, it needs (1, 3) minutes for each of salesman and the cashier, respectively. The veil needs 2 minutes for each of the salesman and the cashier.Knowing that the shop has at least 18 salesmen and at most 14 employees on the sales box.

$$Maximize\ \ Z = 0.5x_1 + 5x_2 + 3x_3$$

subject to $\quad x_1 + \quad x_2 + 2x_3 \geq 18$

$$x_1 + 3x_2 + 2x_3 \leq 14$$

$$x_1, x_2, x_3 \geq 0$$

Cj	Basic Variables	.5 x	5 y	3 Z	0 artfcl 1	0 surplus 1	0 slack 2	Quantity
Iteration 1								
	cj-zj	1	1	2	0	-1	0	
0	artfcl 1	1	1	2	1	-1	0	18
0	slack 2	1	3	2	0	0	1	14
Iteration 2								
	cj-zj	0	-2	0	0	-1	-1	
0	artfcl 1	0	-2	0	1	-1	-1	4
3	Z	0.5	1.5	1	0	0	0.5	7

Table 4-1: Iterations of infeasible solution

The second aspect (geometric aspect): This situation means (infeasible) that there is no solution to the LPP that satisfies the needs of all the constraints, and for the geometric method, this situation occurs when the constraints are in opposite direction, there is no intersection in the feasible solution region.

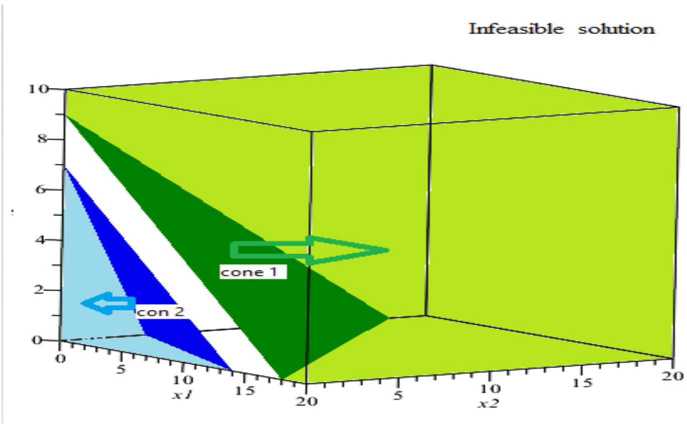

Figure-4-1: *Infeasible solution*

The Third Aspect: Economic and administrative,this situation appears if the problem is formulated incorrectly by placing ineffective restrictions that have nothing to do with production.

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5.Unique Solution:

The Algebraic Aspect: This case appears when solving the problem algebraically,then there is one solution after the trivial solution with a unique point that satisfies all the constraints of the problem.Also, itrepresents the optimal solution.When the simplexmethod is used, the solution is one iteration after the initial solution.

Example 5.1:Back to the Example 4.1 of industrial materials upholstery company and making some changes to it

maximize $Z = 3x_1 + 2x_2 + 7\,x_3$

Subject to$x_1 + \ 2\,x_2 + \ 3\,x_3 \geq 21$

$$x_1 + \quad 3x_2 + 2x_3 \leq 14$$

$$x_3 \leq 7 \qquad , \quad x_1 , \qquad x_2, x_3 \geq 0$$

When used POM-QM software, we found the optimal solution all iterations is

$x_3 = 7$ and the opjective function value $= 49$ that means the solution is unique

Where all the iterations of the solution remain the same $x_3 = 7$,with the objective function still 49.asthe solution appears in Table: 5-1

Cj	Basic Variables	3 X1	2 X2	7 X3	0 artfcl 1	0 surplus 1	0 slack 2	0 slack 3	Quantity
0	artfcl 1	1	2	3	1	-1	0	0	21
0	slack 2	1	3	2	0	0	1	0	14
0	slack 3	0	0	1	0	0	0	1	7
teration 2									
	cj-zj	0	0	0	-1	0	0	0	
7	X3	0.3333	0.6667	1	0.3333	-0.3333	0	0	7
0	slack 2	0.3333	1.6667	0	-0.6667	0.6667	1	0	0
0	slack 3	-0.3333	-0.6667	0	-0.3333	0.3333	0	1	0
teration 3									
	cj-zj	-2.0333	-2.6667	0	-2.3333	2.3333	0	0	
7	X3	0.3333	0.6667	1	0.3333	-0.3333	0	0	7
0	slack 2	0.3333	1.6667	0	-0.6667	0.6667	1	0	0
0	slack 3	-0.3333	-0.6667	0	-0.3333	0.3333	0	1	0
teration 4									
	cj-zj	-3.2	-8.5	0	0	0	-3.5	0	
7	X3	0.5	1.5	1	0	0	0.5	0	7
0	surplus 1	0.5	2.5	0	-1	1	1.5	0	0
0	slack 3	-0.5	-1.5	0	0	0	-0.5	1	0

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Table 5-1:Unique solution

The Second Aspect:when the unique solution is represented by the geometric method with MAPLE software, then the uniquepoint represents the point of intersection of the planes representing the constraints of problem.Figure 6.1 shows this situation

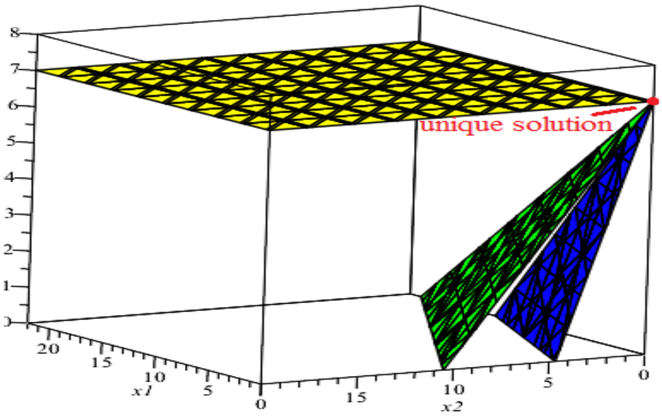

Figure 5-1 : unique solution at one point

The Third Aspect: Economic and administrative meaning that the decision maker does not have many options except this unique solution that produces only third carpet or not produce.

6.conclusions

The three dimensional space of geometrical representations has a huge ability to represent the special cases of solutions. Occurring any special case with three aspects yields a better interpretation to understand such special case. A geometrical representation of three-dimensional space yield a high power ability ofdecision maker to have a best decision and to know obviously the relationship between the constraints of LPP.

7- References

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