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Critical Study of Graph Theory

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Abstract:
Due to its simplicity and wide range of scientific and technical applications, notably in computer science, system theory, and biology, graph theory has recently emerged as a popular area of mathematical study. The mathematical fields of group theory, matrices, numerical analysis, probability, and topology are all intertwined with graph theory. Consequently, algebraic graph theory, and graph theory more generally, have been thrust to the forefront of contemporary practical mathematics.
Keywords: graph theory, mathematics, algebraic, mathematical model

I Introduction:

Graph theory has become an interesting topic of mathematical research because of its simplicity and its scientific and engineering applications especially to computer science, system theory and biology. Graph theory is also closely related to many branches of mathematics including group theory, matrix theory, numerical analysis, probability and topology. For these reasons, graph theory particularly algebraic graph theory has already been accorded a place in the forefront of applied mathematics. Mathematical model of any system involving a binary relation can be represented by graph. Graph theory came into existence during the 1st half of 18th century when Euler (1736) proved an unsolved puzzle of his time known as the Königsberg Bridge Problem. This field of mathematics was in a dormant state until the 2nd half of 19th century. In recent years graph theory has become a well-developed branch of mathematics which has become worthy of study by its own right. It is mentioned earlier that graph theory is closely related to algebra where algebraic techniques can be used to study the properties of graph. We refer to this area of graph theory as “algebraic graph theory”. The literature of algebraic graph theory has grown enormously since 1974. Linear algebra comprising the theory of matrices and linear vector spaces are the most widely applied part of algebra that can be used to study some properties of graphs. The adjacency matrix of a graph completely determines the graph. Another important matrix which completely describes a graph is the incidence matrix of the graph. A symmetry and regularity properties of graphs are reflected by the automorphisms of graph; that is by the permutation of vertices which preserves adjacency. In the chapters of this thesis, we deal with some graph structures which are developed by using some concepts and techniques related to algebra basically ring theory.

The concept of a graph in graph theory has different connotations than the concept of a graph when used to describe a function or statistical data. A non-empty finite collection of objects, or vertices, and a set of unordered (ordered) pairings of vertices of G, or edges, make up a graph G in graph theory (arcs). Both the vertices and edges of a graph G are represented by the notation V(G) and E(G), respectively.

As a discipline, graph theory emerged in the first part of the 18th century. The theory became an extremely useful resource for mathematicians since graphs may be used to depict practically any physical problem requiring discrete groupings of items and a relationship among them. The term "graph" is used to describe a wide variety of mathematical objects that consist of points and the relationships between those points. Before the second part of the 19th century, there had been little progress in the field of graph theory. This theory's systematic growth began in the first part of the twentieth century. The study of leisure issues and games inspired a significant portion of graph theory. Graph theory's versatility is a result of its ease of use in fields as diverse as chemistry, physics, computer science, electrical and civil engineering, architecture, operational research,

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genetics, psychology, sociology, economics, ethnography, and language.

II Graph Theory

In this section we give some basic definitions and results from graph theory that is required in the next chapters. We refer the reader for other basic notations in graphs and hypergraphs.

- **Graphs**

A graph is a pair G = (V (G), E(G)), where V (G) is the vertex set and E(G) is the edge set. Unordered pairs of unique V elements make up the edge set V(G). In this thesis, every G-graph is an elementary one. In other words, a graph does not include any infinite loops or cycles of edges. N (x) represents the collection of nearby vertices to a given vertex x in a graph G. The cardinality of a network N is 2, and the degree of a vertex x in a graph G is the number of vertices of G that are nearby to x, written as deg(x) (x). A graph G is said to be k-regular if and only if every vertex in G has the same degree k.

A walk is a connected set of nodes, where the endpoints of each edge are the nodes that came before and after it in the walk. The number of edges traversed by a stroll is its length. In a graph, a route is a directed walk between any two sets of vertices (except the endpoints). A circle is a closed route that begins and finishes at the same point. A cycle's length is equal to the sum of its vertices. The length of the shortest cycle in a graph G is its girth, or gr(G). The circumference of a graph without cycles is denoted by ∞. A graph must have a minimum girth of 3.

If there is a route connecting every pair of vertices in a graph G, then G is said to be linked. d(x, y) is the length of the shortest path in G that starts at vertex x and ends at vertex y. The maximum distance between any two vertices of G is termed its diameter and is represented by the symbol diam (G). Each vertex in a subset is adjacent to every other vertex in the other subsets, and no edge has both endpoints in any one subset. This is what we mean by "complete" in an r-partite graph. A complete r-partite graph with partitions of size m1, m2, . . . , mr is denoted by Km1,m2,...,mr. The complete bipartite (i.e., 2-partite) graph is denoted by Km,n, where the set of partition has sizes m and n. A complete graph is a graph where each vertex is adjacent to all other vertices. We denote by Kn the complete graph on n vertices.

A clique in a graph G is a maximal complete subgraph of G. The clique number of G is the number of vertices in the largest clique and is denoted by ω(G). A subset S of G is said to be an independent set if no two vertices in S are adjacent. The independent number α(G) is the number of vertices in the largest independent set in G.

A graph H is called a subgraph of a graph G if V (H) ⊆ V (G) and E(H) ⊆ E(G). For any set S of vertices of G, the maximal subgraph of G with vertex set S is called induced subgraph ⟨S⟩ of G induced by S and two vertices in S are adjacent if and only if they are adjacent in G.

Let G and H be two graphs. The corona product G◦ H, is obtained by taking one copy of G and |V (G)| copies of H and by joining each vertex of the i-th copy of H to the i-th vertex of G, where 1 ≤ i ≤ |V (G)|.

An isomorphism of graphs G1 and G2 is a bijection φ between the vertex sets of G1 and G2 such that for any two vertices x and y of G1, x and y are adjacent in G1 if and only if φ(x) and φ(y) are adjacent in G2. If an isomorphism exists between two graphs G1 and G2, then the graphs are called isomorphic and we write G1 ≅ G2.

To put it another way, a split graph is one in which the collection of vertices may be split into a clique and an unrelated set. One way to classify split graphs is by checking for the presence of C4, C5, or 2K2.

- **Hypergraphs**

In order to distinguish hypergraphs from graphs, we use a calligraphy font to denote them, i.e., we write $\mathcal{H}$ = (V, ε) for a hypergraph on the vertex set V and hyperedges ε. We shall call hyperedges simply edges when no confusion arises.

A hypergraph $\mathcal{H}$ is an ordered pair $\mathcal{H}$ = (V, ε), where V is a finite non-empty set (the set of vertices) and ε is a collection of distinct non- empty subsets of V (the set of edges). If all edges have size exactly k, $\mathcal{H}$ is called k-uniform. Thus, a 2-uniform hypergraph is just a graph. A sub-hypergraph $\mathcal{K}$ of $\mathcal{H}$ is a hypergraph such that V ($\mathcal{K}$) ⊆ V ($\mathcal{H}$) and E($\mathcal{K}$) ⊆ E($\mathcal{H}$).

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A hypergraph is said to be linear if any two of its edges inter- sect at most one vertex. A cycle of length q in $\mathcal{H}$ is a sequence (x1, e1, x2, e2, . . . , xq, eq, xq+1) such that q > 1, x1, x2, . . . , xq are all dis- tinct vertices of $\mathcal{H}$, e1, e2, . . . , eq are all distinct edges of $\mathcal{H}$, xk, xk+1 ∈ ek for 1 ≤ k ≤ q and xq+1 = x1. The length of a cycle is the number of edges in it. The girth of a hypergraph is the length of a shortest cycle it contains.

A r-coloring of $\mathcal{H}$ is a mapping c: V (H) → {1, . . . , r} such that no edge of H (besides singletons) has all vertices of the same color. The chromatic number of $\mathcal{H}$, denoted by $\chi(\mathcal{H})$ is the minimal r, for which H admits a r-coloring.

The incidence graph of $\mathcal{H}$ is a bipartite graph I($\mathcal{H}$) with vertex set S = V ∪ ε, and where x ∈ V and e ∈ ε are adjacent if and only if x ∈ e.

III Topological Graph Theory

The arrangement of graphs on surfaces is at the heart of topological graph theory. Using topological concepts to investigate subfields of graph theory, and vice versa, has proven to be a promising area of study. As symmetry becomes increasingly essential in fields like computer networks, there are connections to other branches of mathematics like design theory and geometry.

3.1 Surfaces

Graphs on surfaces form a natural link between discrete and continuous mathematics. A surface is a connected compact Hausdorff topological space $\mathbb{S}$ which is locally homeomorphic to an open disk in the plane (and hence locally homeomorphic to $\mathbb{R}^2$), that is, each point of $\mathbb{S}$ has an open neighborhood homeomorphic to the open unit in $\mathbb{R}^2$.

A surface is orientable if a positive sense of rotation (say, clockwise) can be made around all points consistently, and is non-orientable oth- erwise. The simplest orientable surfaces are the sphere and the torus, while the simplest non-orientable surfaces are the projective plane and the Klein bottle. It is well-known that a compact surface is homeo- morphic to a sphere, a connected sum of g tori, or a connected sum of k projective). The sphere is designated to be the surface $\mathbb{S}$0, the surface formed by g handles to the sphere is denoted $\mathbb{S}$g, and the surface formed dy k mobius strips to the sphere is denoted by $\mathbb{N}$k.

Klein proved that any surface can be represented as a polygon with sides glued together in pairs, but there may be many polygonal representations of the same surface.

Below is a polygonal representation for the double torus.

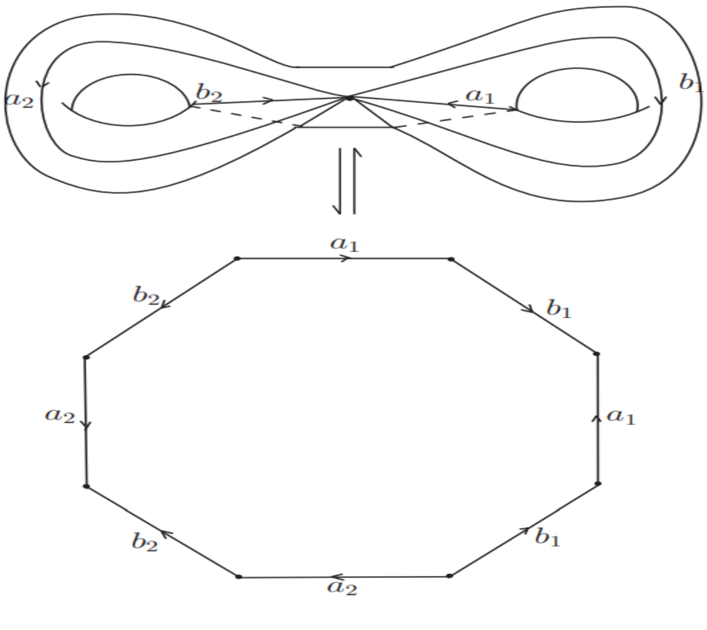

Figure 1. : Polygonal representation of double torus

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3.2 Embeddings

An embedding of a graph G in the plane is a mapping from the plane to the vertices of G, and from the edges of G to disjoint simple open arcs, such that the image of each edge unites the images of its two vertices, and no image of an edge contains the image of a vertex.

Note that one area is unbounded: a region of an embedded graph G is a maximally linked set of points in the relative complement of G in the plane. A face is the topological closure of an area (the region plus the vertices and edges of G on its boundary). For each given face, the length of its associated border walk is equal to the face's size.

A graph embedding is cellular if its faces are homeomorphic to open disks. A graph embedding is a triangulation if it is cellular and its faces have degree three.

The orientable genus of a graph G is the set of integer's g such that the graph G is cellularly embeddable in the surface $\mathbb{S}g$. The minimum of this range is called the genus of the graph. The crosscap (non- orientable genus) of a graph G is the set of integers k such that the graph G is cellularly embeddable in the surface $\mathbb{N}k$. The minimum of this range is called the crosscap of the graph.

For a graph to be considered outerplanar, it must be possible to embed it in the plane such that each of its vertices lies on the same face of the plane. A graph is characterized as outerplanar if and only if it does not contain any subdivisions of K4 or K2,3. A planar graph is a graph that may be embedded on the plane. If a finite graph can embed in the plane, then it can also embed in the sphere. Planar graphs have genus 0, whereas toroidal graphs have genus 1. Graphs with crosscap 1 are considered to be projective if and only if they satisfy this condition.

See an example of K5's toris embedding down below:

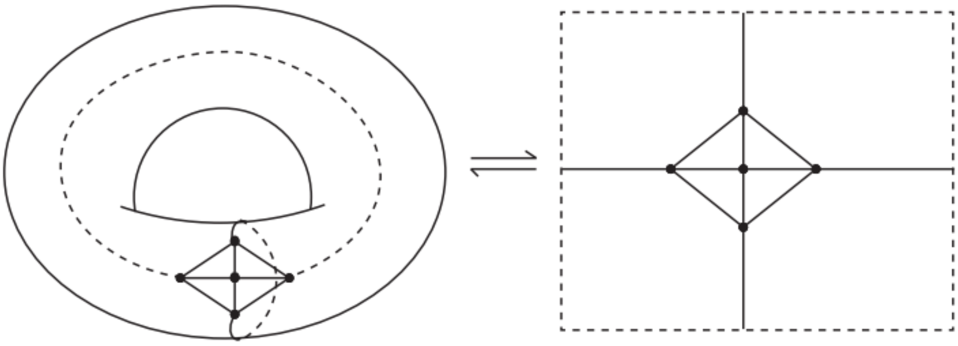

Figure 1.2: Toroidal embedding of K5 in $\mathbb{S}1$

Theorem 1.1: A graph G is planar if and only if it contains no subdivision of K5 or K3,3.

Lemma 1.2: The following statements hold.

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \ if\ n \geq 3.$$

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil \ if\ m, n \geq 2.$$

Lemma 1.3: Let G be a connected graph with n ≥ 3 vertices and q edges. If G contains a cycle of length 3, then

$$\gamma(G) \geq \left\lceil \frac{q}{6} - \frac{n}{2} + 1 \right\rceil$$

Lemma 1.4: Suppose a simple graph G is connected with v ≥ 3 vertices and e edges. If G has no triangles, then

$$\gamma(G) \geq \left\lceil \frac{e}{4} - \frac{v}{2} + 1 \right\rceil$$

Lemma 1.5: If G is a finite connected graph with n vertices, m edges, and genus γ, then n − m + f = 2 − 2γ, where f is the number of faces created when G is minimally embedded on a surface of genus γ.

Lemma 1.6: The following statements hold:

$$\overline{\gamma}(K_n) = \begin{cases} \lceil \frac{1}{6}(n-3)(n-4) \rceil & if\ n \geq 3\ and\ n \neq 7; \\ 3 & if\ n\ =\ 7 \end{cases}$$

$$\overline{\gamma}(K_{m,n}) = \lceil \tfrac{1}{2}(m-2)(n-2) \rceil,\ where\ m, n \geq 2$$

The following Lemma gives us a lower bound for the non-orientable genus of connected graphs.

Lemma 1.7: [26] Let G be a connected graph with $n \geq 3$ vertices and q edges. If G contains a cycle of length 3, then

$$\overline{\gamma}(G) \geq \lceil \tfrac{q}{3} - n + 2 \rceil$$

Lemma 1.8: [26] Let $\varphi : G \to \mathbb{N}k$ be a 2-cell embedding of a connected graph G to the non-orientable surface $\mathbb{N}k$. Then $v-e+f = 2-k$, where v, e and f are the number of vertices, edges, and faces that $\varphi(G)$ has respectively, and k is the crosscap of $\mathbb{N}k$.

Theorem 1.9: [65, Corollary 2] A hypergraph is planar if and only if its incidence graph is planar.

Theorem 1.10: [65, Corollary 1] For any hypergraph $\mathcal{H}$, $\gamma(\mathcal{H}) = \gamma(\mathcal{I}(\mathcal{H}))$.

Theorem 1.11. [65] For any hypergraph H,

$$\overline{\gamma}(\mathcal{H}) = \overline{\gamma}(\mathcal{I}(\mathcal{H})).$$

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