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Applications of Continued Fractions

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Abstract

Many continuing fraction applications from different mathematical contexts and levels are the topic of this investigation. Many continuing fraction qualities are examined as the first step in the investigation. When looking for the best rational approximations of a real number x, the continuous fraction expansion is a particularly effective method. In addition, continuing fractions are a remarkably flexible tool for handling issues involving motions spanning more than one time period. In mathematics, continued fractions play a crucial role. The fact that they may be used in so many different areas of pure and practical research is a major reason for their significance. Although though many people have heard of them, Continuous Fractions (CF) are an old topic. Algebra and other areas of study including arithmetic, physics, and chemistry all make use of repeated fractions.

Keywords: Continued Fraction, Euclidean, Algorithm, Diophantine, Equation

Introduction

The continued fraction is an alternative representation of numbers. Although it is a powerful and insightful representation of numbers, it is largely ignored in the mathematics we have been taught. Every real number may be written as a continuing fraction, which is just the sum of a series of divisions. Many fields have found use for continued fractions. They showed us how to approximate irrational numbers with rational ones. Several computer programs made approximations based on continuous fractions. Solutions to the Diophantine and Pell's equations may also be found via the use of continued fractions. In addition, Robert M. Corless noted in a 1992 study that continuous fractions and chaos theory have some common ground. Applications such as building calendars, astronomy, music, and others all benefit greatly from the use of continuing fractions in the mathematical handling of issues that arise.

There is a lengthy history behind continued fractions, maybe dating back to Euclid's technique for finding the greatest common divisor. Yet, because to their use in modern, fast, and precise computer mathematics, they are enjoying a renaissance. The use of continuing fractions in computer arithmetic has several benefits, including the elimination of roundoff and truncation mistakes, quicker division and multiplication compared to positional number representations, and accurate evaluation of trigonometric, logarithmic, and other functions. In a normal situation, each succeeding numerator would be 1.

The notion of continuing fractions was initially introduced by the Euclidean method for finding the greatest common divisor (GCD) of two numbers, which had been around for quite some time. Around 300 B.C. was the approximate time frame. Since then, research has continued to be conducted, and a plethora of applications have been developed, all thanks to the ease with which these problems can be solved and the speed with which they can be calculated using just addition, subtraction, multiplication, and division. Nevertheless, continuing fractions remain a profitable and attractive area of study. In reality, their applications are evident in a wide variety of fields, including medicine, chemistry, physics, and mathematics.

Literature Review

Han, Guo-Niu (2014) the Jacobi continuous fraction expansion of a power series f may be used to get the Hankel determinants of f. All Hankel determinants of f must, however, be positive if the Jacobi continuing fraction is to exist. We present the "its Hankel continuing fraction". The Hankel continuing fraction is another useful tool for assessing the Hankel determinants. A power series f over a finite field is said to be periodic in the Hankel continuing fraction if and only if it satisfies a quadratic functional equation. As an example of our application, we calculate the Hankel determinant for the standard paperfolding sequence among others. As a consequence, we provide a computer-generated proof of a result proved by Guo, Wu, and Wen that was previously just a conjecture by Coons-Vrbik.

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Sharaf, Mohammed & Saad, A. S. & Motlep, N. (2015) We provide a powerful technique for the universal Y's functions of space dynamics, one that rests on the continuous fractions theory. For every conic motion, the method works (elliptic, parabolic or hyperbolic).

Greene, J. & Schmieg, J. (2017) looked at an extension of traditional continuing fractions in which the "numerator" might be any positive integer other than 1. Next, we generalize even more to the situation when z is a real integer that is not 1. The situation in which z is rational but not an integer is the one on which we concentrate. Substantial discussion is devoted to periodic and n-expansions, with comparisons drawn between the cases in which z is an integer and a rational number. Since z is not an integer, the periodic expansion of n is no longer required. We provide examples of periodic expansions in numerous infinite families.

Continued Fractions

Two non-zero integers, p and q, are assumed to exist. The expression gcd(p, q) represents the largest positive integer d such that:

- d is a divisor of both p and q.
- c ≤d if and only if it is a divisor of both p and q.

Definition 1: If the gcd(p, q) = 1, then the provided numbers p and q are said to be relatively prime.

Theorem 1: Let p, q, and s all be integers. For any x and y ∈Z, if p divides both q and s, then p divides qx + sy.

Algorithm for Division (Theorem 2): A unique pair of positive integers m and r (0 r q) exists such that given two positive numbers p and q (where q > 0), p = q.m + r. The symbol for the dividend is p, the divisor is q, the quotient is m, and the remainder is r.

The Euclidean Algorithm

To determine the largest common factor of two numbers, the Euclidean algorithm may be used. It is made up of a series of subdivisions. This method employs repeated applications of the Division Algorithm until a zero residual is obtained. Given that gcd(p, q) = gcd(±p, ±q), we know that p is greater than q, thus we can conclude that p > q are both positive integers.

Theorem 3: (Euclidean algorithm): Assume that there are two positive numbers p and q, and that p > q. Then, think about the repeated divisions below.

$$
\begin{aligned}
p &= q.a_1 + r_1, \quad 0 < r_1 < q \\
b &= r_1.a_2 + r_2, \quad 0 < r_2 < r_1 \\
r_1 &= r_2.a_3 + r_3, \quad 0 < r_3 < r_2 \\
r_2 &= r_3.a_4 + r_4, \quad 0 < r_4 < r_3 \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
r_{n-2} &= r_{n-1}.a_n + r_n, \quad 0 < r_n < r_{n-1} \\
r_{n-1} &= r_n.a_{n+1} + 0
\end{aligned}
$$

The last non-zero residual after dividing by two is rn, therefore gcd(p, q) = r_n.

Proof: It is necessary to demonstrate that rn is the greatest common divisor of p and q. This is what we get if we apply Lemma 1.1 over and over again:

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$$\gcd(p,\ q)=\gcd(q,\ r_1)=\gcd(r_1,\ r_2)=\gcd(r_2,\ r_3)\ =\ldots=\gcd(r_{n\text{-}1},\ r_n)=\gcd(r_n,\ 0)$$

$$=r_n.$$

Hence, rn is the largest common factor between p and q.

Theorem 4: Given p and q, two non-negative integers, prove Theorem 4. If p and q are linearly independent, then their GCD is a linear combination. That is, gcd(p, q)=mp+nq indicates that m and n are both integers.

The Euclidean algorithm for determining the greatest common divisor of two numbers is the most direct route to proving this. Here's how the algorithm works: Assume a and b are both whole numbers. Given that gcd(a, b) = gcd(b, a) = gcd(|a|, |b|) [Bur11, Ch.2], we know that a ≥ b > 0. If we have a diminishing series of remainders, b > r1 > r2 > ... ≥ 0, using the division procedure numerous times will finally lead to 0. Let's do this the way it's supposed to be done:

$$a = q_1 b + r_1, \quad \text{where } 0 < r_1 < b$$

$$b = q_2 r_1 + r_2, \quad \text{where } 0 < r_2 < r_1$$

$$r_1 = q_2 r_2 + r_3, \quad \text{where } 0 < r_3 < r_2$$

$$\vdots$$

$$r_{n-2} = q_n r_{n-1} + r_n, \quad \text{where } 0 < r_n < r_{n-1}$$

$$r_{n-1} = q_n r_n + 0 \quad [\text{Bur11}, Ch.2].$$

The gcd is the last non-zero residual, rn (a, b). We are less concerned with the gcd itself than with the algorithm used to find it. The FCF form [q1; q2,..., qn] = [a0; a1, a2,..., an] has one element for each quotient (q1, q2,..., qn). Let's try out the algorithm on a few real-world problems. To begin, let's run some numbers through the Euclidean Algorithm with $\frac{187}{57}$ here, a = 187 and b = 57:

187 = (3)57 + 16

57 = (3)16 + 9

16 = (1)9 + 7

9 = (1)7 + 2

7 = (3)2 + 1

2 = (2)1 + 0

The FCF of $\frac{187}{75}$ is [3; 3, 1, 1, 3, 2]. These values may be obtained through the algorithm's quotients. Take note that the gcd(187, 75) equals 1 in this scenario. Consider $\frac{a}{b}$the case where a=147 and b=69 to see still another instance where gcd(a,b)≠ 1.

147 = (2)69 + 9

69 = (7)9 + 6

9 = (1)6 + 3

6 = (2)3 + 0.

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The FCF identifier for

$$\frac{147}{69} = [2; 7, 1, 2] = 2 + \cfrac{1}{7 + \cfrac{1}{1 + \cfrac{1}{\cfrac{1}{2}}}} = \frac{49}{23}$$

This is seen in the image below.

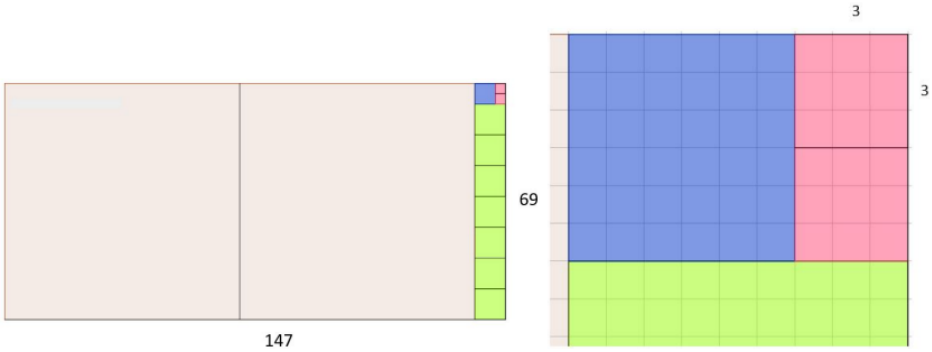

Figure 1: Pictorial Representation of 147/69

To get the tan of a 147 by 69 rectangle, We get the linear residual of 9 by counting the number of 69x69 squares we can generate, and we obtain 2. That's where we get the first pair of numbers in [2; 7; 1]. 2. Given that the first residual is 9, we can next count the number of 9 by 9 green squares that may be made from the remaining material, and we find that there are 7. As a result, we have a linear remaining of 6. Note that the subsequent FCF number is 7. For the next linear remainder of 3, the next digit in the FCF is 1, which we get by counting the number of blue 6x6 squares in the remaining space. The next step is to count the number of pink 3x3 squares that remain, and we find that there are just 2.

Binomial Theorem:

The expansion of $(x + y)^n$ according to the binomial theorem is as follows, for every positive integer n:

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \ldots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

Where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

the coefficient of the binomial distribution, or.

The Algorithm of Continued Fractions

The integral portion of each real number x is represented by a unique integer bxc, while the fractional part is represented by a unique real x in the range [0, 1].

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x = bxc + {x}

In the event that x is not an integer, we get by setting x1:= 1/{x} and {x} ≠ 0.

$$x = \lfloor x \rfloor + \frac{1}{x_1}$$

When we set x2:= 1/{x1} , if {x1} is not an integer, we receive {x1} ≠0.

$$x = \lfloor x \rfloor + \cfrac{1}{\lfloor x_1 \rfloor + \cfrac{1}{x_2}}$$

If for any i {xi} = 0, then the procedure ends; otherwise, it goes on indefinitely. The so-called continuing fraction expansion of x is written a0 := $\lfloor x \rfloor$ and ai = $\lfloor xi \rfloor$ for i ≥ 1.

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

which from now on will be written using the shorter notation

x = [a0, a1, a2, a3, . . .]

Partial quotients of the continuing fraction of x are the integers a0, a1,..., whereas the rational numbers

$$\frac{p_k}{q_k} := [a_0, a_1, a_2, \ldots, a_k]$$

are also known as convergents. The convergents approximate x as closely as possible using just rational functions. If p > q are positive integers, then

$$\left|\frac{p}{q} - x\right| < \frac{1}{2q^2} \quad (1)$$

Thus, p/q converges to x. In fact, if x has two convergents, pk/qk and pk+1/qk+1, then at least one of these fulfills (1).

If x = a/b is a rational integer, then finding its continuing fraction is as simple as using the Euclidean technique to find their greatest common divisor:

$$\begin{array}{lll} a = a_0 b + r_0, & 0 \le r_0 < b, & x_1 = b/r_0, \\ b = a_1 r_0 + r_1, & 0 \le r_1 < r_0, & x_2 = r_0/r_1, \\ r_0 = a_2 r_1 + r_2, & 0 \le r_2 < r_1, & x_3 = r_1/r_2, \\ \ldots & & \end{array}$$

On the one hand, the continuing fraction of a rational integer is always finite since the Euclidean algorithm always halts. In contrast, a finite continuous fraction may be constructed for any rational integer. We have reached this conclusion because the growth of any continuous fraction of a real number is constrained if and only if the number is rational. Don't forget that if ak 2,

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[a0, a1, a2, . . ., ak] = [a0, a1, a2, . . ., ak−1, ak − 1, 1], (2)

Hence, there are at least two methods to write a rational number as a continuing fraction. It turns out that there are only two ways to represent every rational integer as a continuous fraction, and both are supplied by (2).

Elementary Applications of Continued Fractions

One of the many uses of continuing fractions is finding a fraction with a small denominator that approximates a certain probability, percentage, or average rate. As an example, this concept may be used to speculate on the number of "at bats" a baseball player would have had in order to achieve a specific batting average. For calculating a batting average, the ratio of hits to at-bats is divided by three. As 0.334 = $\frac{334}{1000}$, we may safely infer that there were 1000 opportunities for the batter in question. This decimal may be written as $\frac{167}{500}$, and I think it's safe to say that 500 is the absolute minimum number of "at bats" that can be used. But, might I ask, is that right? Is it feasible to have an even lower number of "at bats" than 500 and yet hit 0.334? We can discover it with the aid of continued fractions. Take the rational integer x to the third power, which is 0.334. In such case, x will range from 0.3335 < x < 0.3345. Have a look at the representation of the numbers $\frac{3335}{10000}$ = [0; 2, 1, 666] and $\frac{3345}{10000}$ = [0; 2, 1, 94, 1, 1, 3] in terms of continuing fractions. It follows that we must choose the continuing fraction α such that [0; 2, 1, 666] <α<[0; 2, 1, 94, 1, 1, 3]. The FCFs, 666 in the smaller fraction and 94 in the bigger fraction, are identical up to the point a3. Hence, 94 < a3 < 666 is the FCF of α. We'll use 95 as a3 since 95 94 + 1 = 95, which satisfies our requirement for a rational integer with a small denominator. Let α = [0; 2, 1, 95]. This yields the lowest denominator fraction possible, between $\frac{3335}{10000}$ *and* $\frac{3345}{10000}$:

$$[0;2,1,95] = 0 + \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{95}}} = 0 + \cfrac{1}{2+\cfrac{1}{\frac{96}{95}}} = 0 + \cfrac{1}{2+\frac{95}{96}} = 0 + \cfrac{1}{\frac{287}{96}}$$

$$= \frac{96}{287} \approx 0.3344947735 \approx 0.334$$

That leaves a minimum of 287 probable plate appearances. Let's have a look at an FCF that is more in line with [0; 2, 1, 666], say [0; 2, 1, 600]

$$[0;2,1,600] = 0 + \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{600}}} = 0 + \cfrac{1}{2+\cfrac{1}{\frac{601}{600}}} = 0 + \cfrac{1}{2+\frac{600}{601}} = 0 + \cfrac{1}{\frac{1802}{601}}$$

$$= \frac{601}{1802} \approx 0.333518313 \approx 0.334.$$

In this example, there are 1802 "at bats," which is much more than the usual 287. It is clear that the denominator of the last rational number will be big if the last denominator in the FCF is big where 0.4275 = [0; 2, 2, 1, 18, 3] and 0.4285 = [0; 2, 2, 1, 285]. The range of possible batting averages is [0; 2, 2, 1, 18 3] [0; 2, 2, 1, 285] for the case when y = 0.428. In order to convert [0; 2, 2, 1, 18, 3] ≤ β < [0; 2, 2, 1, 285], we require a continuing fraction β. The smallest of these fractions has an a4 value of 18, while the largest has a value of 285. Because of this, 18 < a4 < 285. In this example, we'll use a4 = 19 and let β= [0; 2, 2, 1, 19] = $\frac{59}{138}$= 0.42753623190 ≈ 0.428. In order to illustrate another characteristic of continuing fractions, we looked at the second case.

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Let's have a look at another use of FCF, this time to determine the lowest feasible denominator rational number that approximates a particular average rate. Opinion surveys are often reported in the media as percentages. It may be said that 70.37 percent of voters approved a measure, for instance. The general public could interpret this to suggest that 7037 out of 10,000 persons voted in favor since this fraction is written in its lowest form: $\frac{7037}{10000}$. Nevertheless, this may not even come close to the reality. Determine the least number of voters needed to get a result of 70.37 percent. The result of 70.37 percent is probably the result of rounding β. This means that the true percentage, which we'll round up to 70.374%, may have been anywhere in that range. The two extreme fractions, $\frac{70365}{100000}$ = [0; 1, 2, 2, 1, 2, 25, 5, 1, 4], and $\frac{70374}{100000}$ = [0; 1, 2, 2, 1, 1, 1, 37, 6, 8], will be used to compute the FCF, just as they were in the batting average problem. Both when a5 = 1 and a5 = 2, these FCF sum to the same value. Let β= [0; 1, 2, 2, 1, 2], where β's a5 is the smaller of the two plus 1. This will get us to a tiny fraction, which is what we need. Then:

$$[0; 1, 2, 2, 1, 2] = 0 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2}}}}} = \frac{19}{27} \approx 0.7037 = 70.37\%$$

This demonstrates that a yes vote of only 19 out of a possible 27 respondents would provide a response rate of 70.37 percent.

Conclusion

This study examines a not-so-typical kind of numerical representation known as continuous fraction. One of the most dramatic and powerful representations of numbers, the continuing fraction has a long and storied history. In the case of irrational numbers, the decimal expansion doesn't always display the full beauty of the underlying numerical pattern. One of the most dramatic and powerful representations of numbers, the continuing fraction has a long and storied history. In the case of irrational numbers, the decimal expansion doesn't always display the full beauty of the underlying numerical pattern. Only quadratic continuing fractions are taken into account.

References

1. Han, Guo-Niu. (2014). Hankel continued fraction and its applications. Advances in Mathematics. 303. 10.1016/j.aim.2016.08.013.
2. Sharaf, Mohammed & Saad, A. S. & Motlep, N. (2015). Continued Fraction Evaluation of the Universal Y's Functions. International Journal of Astronomy and Astrophysics. 5. 15-19. 10.4236/ijaa.2015.51003.
3. Lasjaunias, Alain. (2017). A short history of some recent research on continued fractions in function fields.
4. Greene, J. & Schmieg, J. (2017). Continued fractions with non-integer numerators. 20.
5. D. Roy, On Schmidt and Summerer parametric geometry of numbers, Ann. of Math. (2) 182 (2015), no. 2, 739–786.
6. W. M. Schmidt and L. Summerer, Diophantine approximation and parametric geometry of numbers, Monatsh. Math. 169 (2013), no. 1, 51–104.
7. S. Stadnicki. What are the applications of continued fractions? Mathematics Stack Exchange, 2014. URL:https://math.stackexchange.com/q/766627 (version: 2014-04-23).
8. Ibran, Z.M. (2014) Continued Fractions and Their Applications. M.Sc. Thesis, University of Tripoli, Tripoli.
9. Awin, A.M. (2015) The Use of Special Bilinear Functions in Computing Some Special Functions. Journal of Progressive Research in Mathematics, 5, 437-443.
10. Sharaf, A.M., Saad, A.S. and Abd El Motelp, N.S. (2015) Continued Fraction Evaluation of the Universal Y's Functions. International Journal of Astronomy and Astrophysics, 5, 15-19. http://www.scirp.org/journal/ijaa
https://doi.org/10.4236/ijaa.2015.51003