JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 5499-5513
https://publishoa.com
ISSN: 1309-3452

The Existence of Solution to the Third Order Multiple Delay Differential Equation with Oscillatory Property

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Accepted: 2022 May; November, 16; **Published**: 2022 December, 24

Abstract

The purpose of this paper is to guarantee the existence of solutions which are nonoscillatory by novel conditions for third order multiple delay differential equation (TOMDDE). These conditions succeed to show that the solution is exist with bounded property by using convergent sequences and series. We explained the powerful of novel conditions by illustrative example. The oscillation behavior has been got to the (TOMDDE) by sufficient conditions.

Keywords: Third Order Multiple Delay Differential Equation, Banach Space, Existence of Nonocillatory Bounded Solutions, Oscillatory Behavior.

1. Introduction

The importance of differential equations(DEs) appears by applications in different fields of applied science, engineering, physics and biological models [1-3]. Therefore, it has become important to consider existence of solutions [4,5] and getting approximate or numerical solutions [6,7]. Some authors focused on study qualitative properties such as stability, asymptotic and oscillatory properties for solution [8-10]. Last few years, the delay differential equations emerged in novel models in mathematics and scientific problems, therefore the studying for various types of solutions and their properties has huge interested studies and increasing speedily.

In the present research we have been condensed on thinking about new sufficient conditions to secure the existence of nonoscillatory and bounded solutions with oscillatory property.

In [11] Z. Liu, L. Chen, S. M. Kang, and S. Y. Cho have considered the solvability of a third-order nonlinear neutral delay differential equation of the form:

$$\frac{d}{d\mathfrak{h}}\left(\mathcal{S}(\mathfrak{h})\frac{d}{d\mathfrak{h}}\left(\mathcal{A}(\mathfrak{h})\frac{d}{d\mathfrak{h}}(\mathcal{Z}(\mathfrak{h})+\mathcal{B}(\mathfrak{h})\,\mathcal{Z}(\mathfrak{h}-\tau))\right)\right)+\mathcal{T}\left(\mathfrak{h},\mathcal{Z}\big(\sigma_1(\mathfrak{h}),\mathcal{Z}(\sigma_2(\mathfrak{h})),\dots,\mathcal{Z}(\sigma_n(\mathfrak{h}))\big)\right)=0$$

The Z. Gui in [12] has studied the existence of periodic solutions to the following third-order neutral delay functional differential equation with deviating arguments:

$$\frac{\mathrm{d}^3}{\mathrm{d}\mathfrak{h}^3}\mathcal{Z}(\mathfrak{h})+a\frac{\mathrm{d}^2}{\mathrm{d}\mathfrak{h}^2}\mathcal{Z}(\mathfrak{h})+\mathcal{T}\left(\frac{d}{d\mathfrak{h}}\mathcal{Z}\big(\mathfrak{h}-\tau(\mathfrak{h})\big)\right)+\mathcal{S}\left(\mathcal{Z}\big(\mathfrak{h}-\tau(\mathfrak{h})\big)\right)=\mathcal{A}(\mathfrak{h})$$

O. Moaaz , E. E. Mahmoud and W. R. Alharbi in [13] have obtained a new criterion for the nonexistence of neutral delay differential equations NDDE of third order::

$$\frac{d}{d\mathfrak{h}}\left(\mathcal{A}(\mathfrak{h})\left(\frac{\mathrm{d}^2}{\mathrm{d}\mathfrak{h}^2}\mathcal{T}(\mathfrak{h})\right)^{\alpha}\right)+\mathcal{B}(\mathfrak{h})\mathcal{Z}^{\alpha}(\sigma(\mathfrak{h}))=0,\ \text{where}\ \mathcal{T}(\mathfrak{h})=\mathcal{Z}(\mathfrak{h})+\mathcal{S}(\mathfrak{h})\mathcal{Z}\big(\tau(\mathfrak{h})\big)$$

M.Wei1, C. Jiang and T. Li in [14] have studied the oscillation of the third-order nonlinear neutral differential equations with damping and distributed delay:

$$\frac{d}{d\mathfrak{h}}\left(\mathcal{S}(\mathfrak{h})\frac{d}{d\mathfrak{h}}\left(\mathcal{A}(\mathfrak{h})\left(\mathcal{Z}(\mathfrak{h})+\int_c^d\mathcal{A}(\mathfrak{h},\mu)\,\mathcal{Z}\left(\tau\big((\mathfrak{h},\mu)\big)\right)\right)d\mu\right)\right)+\mathcal{B}(\mathfrak{h})\frac{d}{d\mathfrak{h}}\left(\mathcal{S}(\mathfrak{h})\frac{d}{d\mathfrak{h}}\left(\mathcal{Z}(\mathfrak{h})+\right.\right.$$

$$\left.\left.\int_c^d\mathcal{A}(\mathfrak{h},\mu)\,\mathcal{Z}\left(\tau\big((\mathfrak{h},\mu)\big)\right)d\mu\right)\right)+\int_a^b\mathcal{T}\left(\mathfrak{h},t,\mathcal{Z}\left(M\big((\mathfrak{h},t)\big)\right)\right)dt=0$$

We consider non-linear NDEs with Multiple delays:

$$\frac{\mathrm{d}^3}{\mathrm{d}\mathfrak{h}^3}\mathcal{Z}(\mathfrak{h})=-\sum_{\zeta=1}^{\Gamma}\mathcal{A}_\zeta(\mathfrak{h})\mathcal{T}_\zeta\left(\mathcal{Z}\left(\tau_\zeta(\mathfrak{h})\right)\right)$$

$$+\frac{\mathrm{d}^3}{\mathrm{d}\mathfrak{h}^3}\sum_{\zeta=1}^{\Gamma}\mathcal{B}_\zeta(\mathfrak{h})\mathcal{S}_\zeta\left(\mathfrak{h},\mathcal{Z}\left(\tau_\zeta(\mathfrak{h})\right)\right)\qquad(1.1)$$

During this work we will impose the following hypotheses

(i) $C(H_1,H_2)$ denotes to the set for all functions that are continuous; $f: H_1\to H_2$ with the supremum norm $\|.\|$.

(ii) We suppose that $\mathcal{A}_\zeta$, $\mathcal{B}_\zeta\in \mathrm{C}(\mathfrak{R}^+,\mathfrak{R}^+),(\zeta=1,2,\dots,\eta)$, and the functions $\tau_\zeta:\mathfrak{R}^+\to\mathfrak{R}^+$ are differentiable with $\tau_\zeta(\mathfrak{h})\longrightarrow\infty$ as $\mathfrak{h}\longrightarrow\infty$.

(iii) The functions $\mathcal{T}_\zeta(\mathcal{Z})$ and $\mathcal{S}_\zeta(\mathfrak{h},\mathcal{Z})$ are continuous and satisfy Lipschitz condition in $\mathcal{Z}$. That is, there are positive constants

M_ζ $(\zeta=1,2,\dots,\Gamma)$, such that

$$\left|\mathcal{S}_\zeta(\mathfrak{h},\mathcal{Z})-\mathcal{S}_\zeta(\mathfrak{h},W)\right|\preccurlyeq M_\zeta|\mathcal{Z}-W|\qquad \zeta=1,2,\dots,\Gamma,$$

The solution $\mathcal{Z}(\mathfrak{h})$ satisfy Eq.(1.1) for $\mathfrak{h}\succcurlyeq\mathfrak{h}_1$. We say that solution $\mathcal{Z}(\mathfrak{h})$ is a nonoscillatory solution if it is eventually negative or eventually positive, so there exists $\mathfrak{h}_*\succcurlyeq\mathfrak{h}_0$, such that $\mathcal{Z}(\mathfrak{h})\succ 0$ or $\mathcal{Z}(\mathfrak{h})\prec 0$ for all $\mathfrak{h}\succcurlyeq\mathfrak{h}_*$, otherwise the solution is said to be oscillatory [7].

We need the following lemma and theorem in the main results to second section.

Lemma 1.1: [15] (Theorem to Krasnoselskii of Fixed Point).

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ISSN: 1309-3452

In the space of Banach say X with Ʊ is closed convex bounded set in X, if S_1, S_2: Ʊ ⟶ X, ∋ $S_1 x + S_2 \gamma \in$ Ʊ, $\forall x, \gamma \in$ Ʊ. If S_1 mapping with contractive feature and S_2 is a completely continuous mapping, then $S_1 x + S_2 \gamma = x$ is a solution on Ʊ.

Theorem 1.2 [16] (The Dominated of Convergence to the Lebesgue)

If $\{\wp_n\}$ be sequence to measurable functions on ᴇ. Let q be integrable function on ᴇ with dominates $\{\wp_n\}$ on ᴇ such that $|\wp_n(\chi)| \leqslant q(x)$ on ᴇ, ∀n.If $\{\wp_n\} \to \{\wp\}$ pointwise a.e. on E, then $\wp$ is integrable on ᴇ with:

$\lim\limits_{n\to\infty} \int_{\mathrm{E}} \wp_n = \int_{\mathrm{E}} \wp$, ᴇ is a measurable finite set.

2. Existence of Oscillatory Bounded Solutions:

In this section we introduce new sufficient conditions to ensure the solution is exist and bounded by two positive functions $\mathfrak{L}$ and $\mathfrak{X}$ on $[\mathfrak{h}_1, \infty)$ of Eq. (1.1), $\mathfrak{h}_1 \geqslant \mathfrak{h}_0$. The existence to positive bounded solution has studied while existence of eventually negative solution can be found similarly.

Suppose the following conditions hold in the included results in this section:

A1. $\mathfrak{p}_1 \leqslant \mathcal{A}_\zeta(\mathfrak{h}), \mathcal{B}_\zeta(\mathfrak{h}) \leqslant \mathfrak{p}_2$, $\mathfrak{p}_1, \mathfrak{p}_2 \neq 0$, are constants, $\zeta = 1, 2, 3, \dots, \Gamma$.

A2. $\mathfrak{q}_1 \mathcal{Z}(\mathfrak{h}) \leqslant \mathcal{T}_\zeta\left(\mathcal{Z}\left(\tau_\zeta(\mathfrak{h})\right)\right) \leqslant \mathfrak{q}_2 \mathcal{Z}(\mathfrak{h})$, $\mathfrak{q}_1, \mathfrak{q}_2 \neq 0$, are constants, $\zeta = 1, 2, 3, \dots, \Gamma$.

A3.$\mu_1 \mathcal{Z}(\mathfrak{h}) \leqslant \mathcal{S}_\zeta\left(\mathfrak{h}, \mathcal{Z}\left(\tau_\zeta(\mathfrak{h})\right)\right) \leqslant \mu_2 \mathcal{Z}(\mathfrak{h})$, $\mu_1, \mu_2 \neq 0$, are constants, $\zeta = 1, 2, 3, \dots, \Gamma$.

Theorem 2.1

Assume that A1- A3 hold, and the bounded functions $\mathfrak{L}, \mathfrak{X} \in C^1(\mathbb{N}, [0, \infty))$, and $\mathfrak{h}_1 \geqslant \mathfrak{h}_0 + \rho$, such that

$$\mathfrak{L}(\mathfrak{h}) \leqslant \mathfrak{L}(\mathfrak{h}_1),\ \mathfrak{h}_0 \leqslant \mathfrak{h} \leqslant \mathfrak{h}_1 \tag{2.1}$$

$$\int_{\mathfrak{h}=1}^{\infty} \sum_{\zeta=1}^{\Gamma} \mathfrak{X}\left(\tau_\zeta(t)\right) dt \leqslant \frac{1}{\mathfrak{p}_2 \mu_2}\left(\mathfrak{p}_1 \mathfrak{q}_1 \int_{\mathfrak{h}}^{\infty} \int_{s}^{\infty} \int_{r}^{\infty} \sum_{\zeta=1}^{\Gamma} \mathfrak{L}\left(\tau_\zeta(t)\right) dr\, dt\, ds + \mathfrak{X}(\mathrm{t})\right) \leqslant \mathcal{M}$$

$$\frac{1}{\mathfrak{p}_1 \mu_1}\left(\mathfrak{p}_2 \mathfrak{q}_2 \int_{\mathfrak{h}}^{\infty} \int_{s}^{\infty} \int_{r}^{\infty} \sum_{\zeta=1}^{\Gamma} \mathfrak{X}\left(\tau_\zeta(t)\right) dr\, dt\, ds + \mathfrak{L}(\mathrm{t})\right) \leqslant \int_{\mathfrak{h}=1}^{\infty} \sum_{\zeta=1}^{\Gamma} \mathfrak{L}\left(\tau_\zeta(t)\right) dt, \mathfrak{h} \geqslant \mathfrak{h}_1, \tag{2.2}$$

$$\int_{\mathfrak{h}}^{\infty} \int_{s}^{\infty} \int_{r}^{\infty} \sum_{\zeta=1}^{\Gamma} \mathfrak{X}\left(\tau_\zeta(t)\right) dr\, dt\, ds \prec \infty \tag{2.3}$$

Then the Eq.(1.1) has a bounded solution by positive functions u and v.

Proof

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 5499-5513
https://publishoa.com
ISSN: 1309-3452

Let $I(\mathrm{t}) = \int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathfrak{X}\left(\tau_{\zeta}(t)\right)dr\,dt\,ds$ and then the condition (2.3) implies that

$$\lim_{t\to\infty} I(\mathrm{t}) = \lim_{t\to\infty}\int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathfrak{X}\left(\tau_{\zeta}(t)\right)dr\,dt\,ds = 0. \tag{2.4}$$

Let $(C([\mathfrak{h}_0,\infty),\mathfrak{R}),\|.\|)$ such that $\|\mathcal{Z}\| = \sup_{\mathfrak{h}\geq \mathfrak{h}_0}|\mathcal{Z}(\mathfrak{h})| \Longrightarrow C([\mathfrak{h}_0,\infty),\mathfrak{R})$ is the space of Banach. Let $\mathfrak{D} \subset C([\mathfrak{h}_0,\infty),\mathfrak{R})$ define as:

$$\mathfrak{D} = \{\mathcal{Z}(\mathfrak{h}): \mathcal{Z}(\mathfrak{h}) \in \mathrm{C}([\mathfrak{h}_0,\infty),\mathfrak{R}) \text{ with } \mathfrak{L}(\mathfrak{h}) \preccurlyeq \mathcal{Z}(\mathfrak{h}) \preccurlyeq \mathfrak{X}(\mathfrak{h}), \mathfrak{h} \succcurlyeq \mathfrak{h}_0\} \tag{2.5}$$

Such that $\mathfrak{D}$ is closed and convex.

The mappings Ψ_1 and Ψ_2: $\mathfrak{D} \to C([\mathfrak{h}_0,\infty),\mathfrak{R})$ are defined as:

$$(\Psi_1\mathcal{Z})(\mathfrak{h}) = \begin{cases} \int_{\mathfrak{h}}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathcal{B}_{\zeta}(t)\mathcal{S}_{\zeta}\left(t,\mathcal{Z}\left(\tau_{\zeta}(t)\right)\right)\mathrm{dt} & , \quad \mathfrak{h} \succcurlyeq \mathfrak{h}_1 \\ (\varphi_1\mathcal{Z})(\mathfrak{h}_1), & \quad \mathfrak{h}_0 \preccurlyeq \mathfrak{h} \preccurlyeq \mathfrak{h}_1, \end{cases}$$

$$(\Psi_2\mathcal{Z})(\mathfrak{h}) = \begin{cases} -\int\limits_{\mathfrak{h}}^{\infty}\int\limits_{s}^{\infty}\int\limits_{r}^{\infty}\sum\limits_{\zeta=1}^{\Gamma}\mathcal{A}_{\zeta}(t)\mathcal{T}_{\zeta}\left(\mathcal{Z}\left(\tau_{\zeta}(t)\right)\right)dz\,dt\,ds & , \mathfrak{h} \succcurlyeq \mathfrak{h}_1, \\ (\varphi_2\mathcal{Z})(\mathfrak{h}_1) - \mathfrak{L}(\mathfrak{h}_1) + \mathfrak{X}(\mathfrak{h}) & , \mathfrak{h}_0 \preccurlyeq \mathfrak{h} \preccurlyeq \mathfrak{h}_1, \end{cases} \tag{2.6}$$

Where Ψ_1 and Ψ_2 satisfies eq (1.1)

For all $\mathcal{Z}, W \in \mathfrak{D}$ and $\mathfrak{h} \succcurlyeq \mathfrak{h}_1$, then:

$$\begin{aligned}(\Psi_1 Z)(\mathfrak{h}) + (\Psi_2 W)(\mathfrak{h}) &= \int\limits_{\mathfrak{h}}^{\infty}\sum\limits_{\zeta=1}^{\Gamma}\mathcal{B}_{\zeta}(\xi)\mathcal{S}_{\zeta}\left(t,\mathcal{Z}\left(\tau_{\zeta}(t)\right)\right)dt - \int\limits_{\mathfrak{h}}^{\infty}\int\limits_{s}^{\infty}\int\limits_{r}^{\infty}\sum\limits_{\zeta=1}^{\Gamma}\mathcal{A}_{\zeta}(t)\mathcal{T}_{\zeta}\left(W\left(\tau_{\zeta}(t)\right)\right)dz\,dt\,ds \\ &\preccurlyeq \mathfrak{p}_2\mu_2\int\limits_{\mathfrak{h}}^{\infty}\sum\limits_{\zeta=1}^{\Gamma}\mathcal{Z}\left(\tau_{\zeta}(\mathrm{t})\right)\mathrm{dt} - \mathfrak{p}_1\mathfrak{q}_1\int\limits_{\mathfrak{h}}^{\infty}\int\limits_{s}^{\infty}\int\limits_{r}^{\infty}\sum\limits_{\zeta=1}^{\Gamma}W\left(\tau_{\zeta}(t)\right)dz\,dt\,ds \\ &\preccurlyeq \mathfrak{p}_2\mu_2\int\limits_{\mathfrak{h}}^{\infty}\sum\limits_{\zeta=1}^{\Gamma}\mathfrak{X}\left(\tau_{\zeta}(\mathrm{t})\right)\mathrm{dt} - \mathfrak{p}_1\mathfrak{q}_1\int\limits_{\mathfrak{h}}^{\infty}\int\limits_{s}^{\infty}\int\limits_{r}^{\infty}\sum\limits_{\zeta=1}^{\Gamma}\mathfrak{L}\left(\tau_{\zeta}(t)\right)dz\,dt\,ds \\ &\preccurlyeq \mathfrak{p}_2\mu_2\int\limits_{\mathfrak{h}}^{\infty}\sum\limits_{\zeta=1}^{\Gamma}\mathfrak{X}\left(\tau_{\zeta}(\mathrm{t})\right)\mathrm{dt} - \mathfrak{p}_2\mu_2\int\limits_{\mathfrak{h}}^{\infty}\sum\limits_{\zeta=1}^{\Gamma}\mathfrak{X}\left(\tau_{\zeta}(\mathrm{t})\right) + \mathfrak{X}(\xi) = \mathfrak{X}(\xi)\end{aligned}$$

$\forall\mathfrak{h} \in [\mathfrak{h}_0,\mathfrak{h}_1]$, we have

$$\begin{gathered}(\Psi_1\mathcal{Z})(\mathfrak{h}) + (\Psi_2 W)(\mathfrak{h}) = (\Psi_1\mathcal{Z})(\mathfrak{h}_1) + (\Psi_2 W)(\mathfrak{h}_1) - \mathfrak{L}(\mathfrak{h}_1) + \mathfrak{X}(\mathfrak{h}) \\ \preccurlyeq \mathfrak{X}(\mathfrak{h}_1) - \mathfrak{L}(\mathfrak{h}_1) + \mathfrak{X}(\mathfrak{h}) \preccurlyeq \mathfrak{L}(\mathfrak{h}_1) - \mathfrak{L}(\mathfrak{h}_1) + \mathfrak{X}(\mathfrak{h}) \preccurlyeq \mathfrak{X}(\mathfrak{h}).\end{gathered}$$

So, $\forall\,\mathfrak{h} \succcurlyeq \mathfrak{h}_1$, yield :

$$(\Psi_1\mathcal{Z})(\mathfrak{h})+(\Psi_2 W)(\mathfrak{h})$$

$$=\int_{\mathfrak{h}}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathcal{B}_\zeta(\mathfrak{h})\mathcal{S}_\zeta\left(t,\mathcal{Z}\left(\tau_\zeta(t)\right)\right)dt-\int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathcal{A}_\zeta(t)\mathcal{T}_\zeta\left(W\left(\tau_\zeta(t)\right)\right)dz\,dt\,ds$$

$$\geqslant \mathfrak{p}_1\mu_1\int_{\mathfrak{h}}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathcal{Z}\left(\tau_\zeta(\mathrm{t})\right)\mathrm{dt}-\mathfrak{p}_2\mathfrak{q}_2\int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma}W\left(\tau_\zeta(t)\right)dz\,dt\,ds$$

$$\geqslant \mathfrak{p}_1\mu_1\int_{\mathfrak{h}}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathfrak{L}\left(\tau_\zeta(\mathrm{t})\right)\mathrm{dt}-\mathfrak{p}_2\mathfrak{q}_2\int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathfrak{X}\left(\tau_\zeta(t)\right)dz\,dt\,ds$$

$$\geqslant \mathfrak{p}_1\mu_1\int_{\mathfrak{h}}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathfrak{L}\left(\tau_\zeta(\mathrm{t})\right)\mathrm{dt}-\mathfrak{p}_1\mu_1\int_{\mathfrak{h}}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathfrak{L}\left(\tau_\zeta(\mathrm{t})\right)\mathrm{dt}+\mathfrak{L}(\mathfrak{h})=\mathfrak{L}(\mathfrak{h})$$

$\forall\ \mathfrak{h}\in[\mathfrak{h}_0,\mathfrak{h}_1]$, from Eq. (2.2), we secure:

$$(\Psi_1\mathcal{Z})(\mathfrak{h})+(\Psi_2 W)(\mathfrak{h})=(\Psi_1\mathcal{Z})(\mathfrak{h}_1)+(\Psi_2 W)(\mathfrak{h}_1)-\mathfrak{X}(\mathfrak{h}_1)+\mathfrak{L}(\mathfrak{h})$$
$$\geqslant \mathfrak{L}(\mathfrak{h}_1)-\mathfrak{L}(\mathfrak{h}_1)+\mathfrak{X}(\mathfrak{h})=\mathfrak{X}(\mathfrak{h})\geqslant \mathfrak{L}(\mathfrak{h}) \tag{2.7}$$

So, $\Psi_1\mathcal{Z}+\Psi_2 W\in\mathfrak{D},\forall\ \mathcal{Z},W\in\mathfrak{D},\mathcal{Z}>W$. Now, we have to prove that Ψ_1 is contraction mapping on $\mathfrak{D}$. $\forall\ \mathcal{Z},W\in\mathfrak{D}$ for $\mathfrak{h}\geqslant\mathfrak{h}_1$:

$$\|\Psi_1\mathcal{Z}-\Psi_1 W\|=\sup_{t\geq t_1}|(\Psi_1\mathcal{Z})(\xi)-(\Psi_1 W)(\xi)|$$

$$=\sup_{\mathfrak{h}\geqslant\mathfrak{h}_1}\left|\int_{\mathfrak{h}}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathcal{B}_\zeta(\mathrm{t})\mathcal{S}_\zeta\left(\mathrm{t},\mathcal{Z}\left(\tau_\zeta(\mathrm{t})\right)\right)dt-\int_{\mathfrak{h}}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathcal{B}_\zeta(\mathrm{t})\,\mathcal{S}_\zeta\left(\mathrm{t},W\left(\tau_\zeta(\mathrm{t})\right)\right)dt\right|$$

$$\|\Psi_1\mathcal{Z}-\Psi_1 W\|\leqslant\sup_{\mathfrak{h}\geqslant\mathfrak{h}_1}\left|\mathfrak{p}_2\,\mu_2\int_{\mathfrak{h}=1}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathcal{Z}\left(\tau_\zeta(t)\right)dt-\ \mathfrak{p}_2\,\mu_2\int_{\mathfrak{h}}^{\infty}\sum_{\zeta=1}^{\Gamma}W\left(\tau_\zeta(t)\right)dt\right|$$

$$\leqslant\sup_{\mathfrak{h}\geqslant\mathfrak{h}_1}\left|\mathfrak{p}_2\,\mu_2\int_{\mathfrak{h}=1}^{\infty}\sum_{\zeta=1}^{\lambda}\mathfrak{X}\left(\tau_\zeta(t)\right)dt-\mathfrak{p}_2\,\mu_2\int_{\mathfrak{h}}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathfrak{L}\left(\tau_\zeta(t)\right)dt\right|$$

By condition (2.1) we have

$$\leqslant\sup_{\mathfrak{h}\geqslant\mathfrak{h}_1}\left|\mathfrak{p}_2\,\mathcal{M}\mathfrak{q}_2+\frac{1}{\sigma_2}\mathfrak{X}(\mathfrak{h})-\mathfrak{p}_2\,\mathcal{M}\mathfrak{q}_2-\frac{1}{\sigma_2}\mathfrak{L}(\mathfrak{h})\right|$$
$$\leqslant\sup_{\mathfrak{h}\geqslant\mathfrak{h}_1}\left|\frac{1}{\mathfrak{p}_2}\mathfrak{X}(\mathfrak{h})-\frac{1}{\mathfrak{p}_2}\mathfrak{L}(\mathfrak{h})\right|$$
$$\leqslant\sup_{\mathfrak{h}\geqslant\mathfrak{h}_1}\frac{1}{\mathfrak{p}_2}|\mathfrak{X}(\mathfrak{h})-\mathfrak{L}(\mathfrak{h})|\leqslant\sup_{\mathrm{t}\geqslant\mathrm{t}_1}\frac{1}{\mathfrak{p}_2}|\mathcal{Z}(\mathfrak{h})-W(\mathfrak{h})|)$$

$$\leqslant M\|\mathcal{Z} - W\| \tag{2.8}$$

Where , $M = \frac{1}{\mathfrak{p}_2}$

Also for $\mathfrak{h} \in [\mathfrak{h}_0, \mathfrak{h}_1]$.

$$\|\Psi_1\mathcal{Z} - \Psi_1 W\| = \sup_{\mathfrak{h}_0 \leqslant \mathfrak{h} \leqslant \mathfrak{h}_1} |(\Psi_1\mathcal{Z})(\mathfrak{h}) - (\Psi_1 W)(\mathfrak{h})|$$

$$= \sup_{\mathfrak{h}_0 \leqslant \mathfrak{h} \leqslant \mathfrak{h}_1} |(\Psi_1\mathcal{Z})(\mathfrak{h}_1) - (\Psi_1 W)(\mathfrak{h}_1)|$$

$$= \sup_{\mathfrak{h} \geqslant \mathfrak{h}_1} \left| \int_{\mathfrak{h}_1}^{\infty} \sum_{\zeta=1}^{\Gamma} \mathcal{B}_\zeta(\mathrm{t}) \mathcal{S}_\zeta\left(\mathrm{t}, \mathcal{Z}\left(\tau_\zeta(t)\right)\right) dt - \int_{\mathfrak{h}}^{\infty} \sum_{\zeta=1}^{\Gamma} \mathcal{B}_\zeta(\mathrm{t})\, \mathcal{S}_\zeta\left(\mathrm{t}, W\left(\tau_\zeta(\mathrm{t})\right)\right) dt \right|$$

$$\leqslant \left| \mathfrak{p}_2\, \mu_2 \int_{\mathfrak{h}_1=1}^{\infty} \sum_{\zeta=1}^{\Gamma} \mathcal{Z}\left(\tau_\zeta(t)\right) dt - \mathfrak{p}_2\, \mu_2 \int_{\mathfrak{h}_1}^{\infty} \sum_{\zeta=1}^{\Gamma} W\left(\tau_\zeta(t)\right) dt \right|$$

$$\leqslant \left| \mathfrak{p}_2\, \mu_2 \int_{\mathfrak{h}_1=1}^{\infty} \sum_{\zeta=1}^{\Gamma} \mathfrak{X}\left(\tau_\zeta(t)\right) dt - \mathfrak{p}_2\, \mu_2 \int_{\mathfrak{h}_1=1}^{\infty} \sum_{\zeta=1}^{\Gamma} \mathfrak{L}\left(\tau_\zeta(t)\right) dt \right|$$

By condition (2.1) we have

$$\leqslant \left| \mathfrak{p}_2\, \mathcal{M}\mathfrak{q}_2 + \frac{1}{\sigma_2}\mathfrak{X}(\mathfrak{h}_1) - \mathfrak{p}_2\, \mathcal{M}\mathfrak{q}_2 - \frac{1}{\sigma_2}\mathfrak{L}(\mathfrak{h}_1) \right|$$

$$= \left| \frac{1}{\mathfrak{p}_2}\mathfrak{X}(\mathfrak{h}) - \frac{1}{\mathfrak{p}_2}\mathfrak{L}(\mathfrak{h}) \right|$$

$$= \sup_{\mathfrak{h} \geqslant \mathfrak{h}_1} \frac{1}{\mathfrak{p}_2} |\mathfrak{X}(\mathfrak{h}) - \mathfrak{L}(\mathfrak{h})| \leqslant \sup_{\mathrm{t} \geqslant \mathrm{t}_1} \frac{1}{\mathfrak{p}_2} |\mathcal{Z}(\mathfrak{h}) - W(\mathfrak{h})|)$$

$$\leqslant M\|\mathcal{Z} - W\| \tag{2.9}$$

Where , $M = \frac{1}{\mathfrak{p}_2}$ This implies that

$$\|\Psi_1\mathcal{Z} - \Psi_1 W\| \leqslant M\|\mathcal{Z} - W\| \tag{2.10}$$

Thus, Ψ_1 is mapping with contractive property on $\mathfrak{D}$. Now, we have to prove that Ψ_2 has completely property to continuous mapping. First of all, we need to show that Ψ_2 is continuous mapping.

Let $\mathcal{Z}_k = \mathcal{Z}_k(\mathfrak{h}) \in \mathfrak{D}$. Since $\mathfrak{D}$ is closed, thus $\mathcal{Z}_k(\mathfrak{h})$ tend to $\mathcal{Z}(\mathfrak{h})$ as $\mathfrak{h} \to \infty$, $\mathcal{Z}(\mathfrak{h}) \in \mathfrak{D}$. For $\mathfrak{h} \geqslant \mathfrak{h}_1$, yield:

$$\|(\Psi_2\mathcal{Z}_k)(\mathfrak{h}) - (\Psi_2\mathcal{Z})(\mathfrak{h})\| = \sup_{\mathfrak{h} \geqslant \mathfrak{h}_1} |(\Psi_2\mathcal{Z}_k)(\mathfrak{h}) - (\Psi_2\mathcal{Z})(\mathfrak{h})|$$

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 5499-5513
https://publishoa.com
ISSN: 1309-3452

$$\preccurlyeq \sup_{\mathfrak{h} \geqslant \mathfrak{h}_1} \left| -\int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma} \mathcal{A}_{\zeta}(t)\mathcal{T}_{\zeta}\left(\mathcal{Z}_k\left(\tau_{\zeta}(t)\right)\right) dz\, dt\, ds + \int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma} \mathcal{A}_{\zeta}(t)\mathcal{T}_{\zeta}\left(\mathcal{Z}\left(\tau_{\zeta}(t)\right)\right) dz\, dt\, ds \right|$$

$$\preccurlyeq \sup_{\mathfrak{h} \geqslant \mathfrak{h}_1} \mathfrak{p}_2 \left| -\int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma} \mathcal{T}_{\zeta}\left(\mathcal{Z}_k\left(\tau_{\zeta}(t)\right)\right) dz\, dt\, ds + \int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma} \mathcal{T}_{\zeta}\left(\mathcal{Z}\left(\tau_{\zeta}(t)\right)\right) dz\, dt\, ds \right|$$

$$\preccurlyeq \sup_{\mathfrak{h} \geqslant \mathfrak{h}_1} \mathfrak{p}_2 \left| \int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma} \mathcal{T}_{\zeta}\left(\mathcal{Z}_k\left(\tau_{\zeta}(t)\right)\right) dz\, dt\, ds - \int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma} \mathcal{T}_{\zeta}\left(\mathcal{Z}\left(\tau_{\zeta}(t)\right)\right) dz\, dt\, ds \right|$$

$$\preccurlyeq \sup_{\mathfrak{h} \geqslant \mathfrak{h}_1} \mathfrak{p}_2\mathfrak{q}_2 \left| \int_{\mathfrak{n}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma} \mathcal{Z}_k\left(\tau_{\zeta}(t)\right) dz\, dt\, ds - \int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma} \mathcal{Z}\left(\tau_{\zeta}(t)\right) dz\, dt\, ds \right|$$

$$\preccurlyeq \sup_{\mathfrak{h} \geqslant \mathfrak{h}_1} \mathfrak{p}_2\mathfrak{q}_2 \left| \int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma} \mathfrak{X}_k\left(\tau_{\zeta}(t)\right) dz\, dt\, ds - \int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma} \mathfrak{X}\left(\tau_{\zeta}(t)\right) dz\, dt\, ds \right|$$

$$\preccurlyeq \sup_{\mathfrak{h} \geqslant \mathfrak{h}_1} \mathfrak{p}_2\mathfrak{q}_2 \left| \int_{\mathfrak{h} \geqslant \mathfrak{h}_1}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma} [\mathfrak{X}_k\left(\tau_{\zeta}(t)\right) - \mathfrak{X}\left(\tau_{\zeta}(t)\right)] dz\, dt\, ds \right|$$

$$\preccurlyeq \sup_{\mathfrak{h} \geqslant \mathfrak{h}_1} \mathfrak{p}_2\mathfrak{q}_2 \Bigg(\left| \int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty} [\mathfrak{X}_k\left(\tau_1(t)\right) - \mathfrak{X}\left(\tau_1(t)\right)] dz\, dt\, ds \right| + \left| \int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty} [\mathfrak{X}_k\left(\tau_2(t)\right) - \mathfrak{X}\left(\tau_2(t)\right)] dz\, dt\, ds \right| + \cdots + \left| \int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty} [\mathfrak{X}_k\left(\tau_{\Gamma}(t)\right) - \mathfrak{X}\left(\tau_{\zeta}(t)\right)] dz\, dt\, ds \right| \Bigg)$$

According to (2.3), and the bounded property of $\left(\mathfrak{X}\left(\tau_{\zeta}(\mathfrak{h})\right)\right)$ we get

$$\int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathcal{Z}\left(\tau_\zeta(t)\right)dz\,dt\,ds \prec \infty \quad , \tag{2.12}$$

Since $\left|\mathfrak{X}_k\left(\tau_\zeta(\mathfrak{h})\right)-\mathfrak{X}\left(\tau_\zeta(\mathfrak{h})\right)\right| \to 0$, as k tend to ∞, $\zeta = 1, 2, 3, \dots, \Gamma$. By dominative convergence theorem to Lebesgue, yield:

$$\lim_{k\to\infty}\|(\Psi_2\mathcal{Z}_k)(\mathfrak{h})-(\Psi_2\mathcal{Z})(\mathfrak{h})\| = 0 \tag{2.13}$$

It reduces that Ψ_2 will be continuous mapping.

To prove that $\Psi_2\mathfrak{D}$ is a relatively compact, we must accentual that $\{\Psi_2\mathcal{Z} : \mathcal{Z} \in \mathfrak{D}\}$ is uniformly bounded and equicontinuous on $[\mathfrak{h}_0, \infty]$, by theorem of Arzelã-Ascoli [17]. From (2.5), yield $\{\Psi\mathcal{Z} : \mathcal{Z} \in \mathfrak{D}\}$ is a uniformly bounded.

To secure that $\{\Psi_2\mathcal{Z} : \mathcal{Z} \in \mathfrak{D}\}$ is equicontinuouson on $[\mathfrak{h}_0, \infty)$, let $\mathcal{Z} \in \mathfrak{D}$ and any $\varepsilon \succ 0$, by (2.12), so $\exists\ \mathfrak{h}_* \succcurlyeq \mathfrak{h}_1$ large enough:

$$\int_{\mathfrak{h}_*}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathcal{Z}\left(\tau_\zeta(t)\right)dz\,dt\,ds \prec \frac{\varepsilon}{2\mathfrak{q}_2\sigma_2}, \quad ,$$

$$\mathfrak{h} \succcurlyeq \mathfrak{h}_* \succcurlyeq \mathfrak{h}_1 \quad , \tag{2.14}$$

Then, for any given $\varepsilon \succ 0$ and $\mathcal{Z} \in \Psi$, $T_2 \succ T_1 \succcurlyeq \mathfrak{h}_*$, we have

$$\begin{aligned}
\|(\Psi_2\mathcal{Z}_k)(T_2)-(\Psi_2\mathcal{Z})(T_1)\| &= \sup_{T_2\succ T_1\succcurlyeq \mathfrak{t}_*}|(\Psi_2\mathcal{Z}_k)(T_2)-(\Psi_2\mathcal{Z})(T_1)| \\
&\preccurlyeq |(\Psi_2\mathcal{Z}_k)(T_2)| + |(\Psi_2\mathcal{Z})(T_1)|
\end{aligned}$$

$$\begin{aligned}
&\preccurlyeq \int_{T_2}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathcal{A}_\zeta(t)\mathcal{T}_\zeta\left(\mathcal{Z}_k\left(\tau_\zeta(t)\right)\right)dz\,dt\,ds + \int_{T_1}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathcal{A}_\zeta(t)\mathcal{T}_\zeta\left(\mathcal{Z}\left(\tau_\zeta(t)\right)\right)dz\,dt\,ds \\
&\preccurlyeq \mathfrak{p}_2\int_{T_2}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathcal{T}_\zeta\left(\mathcal{Z}_k\left(\tau_\zeta(t)\right)\right)dz\,dt\,ds + \mathfrak{p}_2\int_{T_1}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathcal{T}_\zeta\left(\mathcal{Z}\left(\tau_\zeta(t)\right)\right)dz\,dt\,ds \\
&\preccurlyeq \mathfrak{p}_2\mathfrak{q}_2\int_{T_2}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathcal{Z}_k\left(\tau_\zeta(t)\right)dz\,dt\,ds + \mathfrak{p}_2\mathfrak{q}_2\int_{T_1}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathcal{Z}\left(\tau_\zeta(t)\right)dz\,dt\,ds \\
&\preccurlyeq \mathfrak{p}_2\mathfrak{q}_2\int_{T_2}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathfrak{X}_k\left(\tau_\zeta(t)\right)dz\,dt\,ds + \mathfrak{p}_2\mathfrak{q}_2\int_{T_1}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathfrak{X}\left(\tau_\zeta(t)\right)dz\,dt\,ds
\end{aligned}$$

$$\prec \mathfrak{q}_2\mathfrak{p}_2\frac{\varepsilon}{2\mathfrak{q}_2\mathfrak{p}_2} + \mathfrak{q}_2\mathfrak{p}_2\frac{\varepsilon}{2\mathfrak{q}_2\mathfrak{p}_2} = \varepsilon, \tag{2.15}$$

For $\mathcal{Z} \in \mathfrak{D}$ and $\xi_1 \preccurlyeq T_1 \prec T_2 \preccurlyeq \mathfrak{h}_*$, we get

$$\|(\Psi_2\mathcal{Z})(T_2)-(\Psi_2\chi)(T_1)\| = \sup_{\mathfrak{h}_1\preccurlyeq T_1\prec T_2\preccurlyeq \mathfrak{h}_*}|(\Psi_2\mathcal{Z})(T_2)-(\Psi_2\mathcal{Z})(T_1)|$$

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https://publishoa.com
ISSN: 1309-3452

$$= \sup_{\mathfrak{h}_1 \preccurlyeq T_1 \prec T_2 \preccurlyeq \mathfrak{h}_*} \left| \int_{T_1}^{t_*} \int_s^{\infty} \int_r^{\infty} \sum_{\zeta=1}^{\Gamma} \mathcal{A}_\zeta(t) \mathcal{T}_\zeta\left(\mathcal{Z}\left(\tau_\zeta(t)\right)\right) dz\, dt\, ds - \int_{T_2}^{t_*} \int_s^{\infty} \sum_{\zeta=1}^{\Gamma} \mathcal{A}_\zeta(t) \mathcal{T}_\zeta\left(\mathcal{Z}\left(\tau_\zeta(t)\right)\right) dt\, ds \right|$$

$$\preccurlyeq \sup_{\mathfrak{h}_1 \preccurlyeq T_1 \prec T_2 \preccurlyeq \mathfrak{h}_*} \mathfrak{p}_2 \left| \int_{T_1}^{t_*} \int_s^{\infty} \int_r^{\infty} \sum_{\zeta=1}^{\Gamma} \mathcal{T}_\zeta\left(\mathcal{Z}\left(\tau_\zeta(t)\right)\right) dz\, dt\, ds - \int_{T_2}^{t_*} \int_s^{\infty} \sum_{\zeta=1}^{\Gamma} \mathcal{T}_\zeta\left(\mathcal{Z}\left(\tau_\zeta(t)\right)\right) dt\, ds \right|$$

$$= \mathfrak{p}_2 \int_{T_1}^{T_2} \int_s^{\infty} \int_r^{\infty} \sum_{\zeta=1}^{\Gamma} \mathcal{T}_\zeta\left(\mathcal{Z}\left(\tau_\zeta(t)\right)\right) dz\, dt\, ds$$

$$\preccurlyeq \mathfrak{p}_2 \mathfrak{q}_2 \int_{T_1}^{T_2} \int_s^{\infty} \int_r^{\infty} \sum_{\zeta=1}^{\Gamma} \mathcal{Z}\left(\tau_\zeta(t)\right) dz\, dt\, ds$$

$$\preccurlyeq \mathfrak{p}_2 \mathfrak{q}_2 \frac{\varepsilon}{2\mathfrak{q}_2 \mathfrak{p}_2} (T_2 - T_1).$$

Thus there exists $\delta_1 = \frac{2}{\sqrt{\varepsilon}}$, such that

$$|(\Psi_2 \mathcal{Z})(T_2) - (\Psi_2 \mathcal{Z})(T_1)| \prec \varepsilon_1, \text{if } \ 0 \prec T_2 - T_1 \prec \delta_1, \text{ and } \sqrt{\varepsilon} = \varepsilon_1 \tag{2.16}$$

Finally, let $F(\mathfrak{h}) = \frac{\mathfrak{X}(\mathfrak{h})}{\mathcal{A}(\mathfrak{h})}$, then for any $\mathcal{Z} \in \Psi$, $\mathfrak{h}_0 \preccurlyeq T_1 \prec T_2 \preccurlyeq \mathfrak{h}_1$, by mean value theorem there exist $k_1 \in (T_1, T_2)$ and $\delta_2 = \frac{\varepsilon}{F'(k_1)} \succ 0$ such that

$$\begin{aligned} |(\Psi_2 \mathcal{Z})(T_2) - (\Psi_2 \mathcal{Z})(T_1)| &= \left| \left(\frac{\mathfrak{X}}{\mathcal{A}}\right)(T_2) - \left(\frac{\mathfrak{X}}{\mathcal{A}}\right)(T_1) \right| \\ &= |F(T_2) - F(T_1)| \\ &= |F'(k_1)(T_2 - T_1)| \\ &= |F'(k_1)|(T_2 - T_1) \prec \varepsilon, \\ &\text{if } \ 0 \prec T_2 - T_1 \prec \delta_2 \prec \delta_1. \end{aligned} \tag{2.17}$$

Hence $\Psi_2 \mathfrak{D}$ is a compact relatively set. By using lemma (1.1), it reduces that Eq. (1.1) has solution which is bounded relatively from below.

Example 2.1

$$\frac{d^3}{d\xi^3} \mathcal{Z}(\mathfrak{h}) = -\sum_{\zeta=1}^{\Gamma} \mathcal{A}_\zeta(\mathfrak{h}) \mathcal{T}_\zeta\left(\mathcal{Z}\left(\tau_\zeta(\mathfrak{h})\right)\right) + \frac{d^2}{d\xi^2} \sum_{\zeta=1}^{\Gamma} \mathcal{B}_\zeta(\mathfrak{h}) \mathcal{S}_\zeta\left(\mathfrak{h}, \mathcal{Z}\left(\tau_\zeta(\mathfrak{h})\right)\right)$$

Set $\mathfrak{p}_1 = \mu_1 = \mathfrak{q}_{1=1}$, $\mathfrak{p}_2 = \mu_2 = \mathfrak{q}_2 = 2, \tau_\varsigma(\mathfrak{h}) = \mathfrak{h} + 1, \mathfrak{X}(\mathfrak{h}) = \frac{15}{\mathfrak{h}^4}, \mathfrak{L}(\mathfrak{h}) = \frac{12}{\mathfrak{h}^4}, \mathfrak{X}\left(\tau_\zeta(\mathfrak{h})\right) = \frac{15}{(\mathfrak{h}+1)^4}, \mathfrak{L}\left(\tau_\zeta(\mathfrak{h})\right) = \frac{12}{(\mathfrak{h}+1)^4}, \zeta = 1,2$

$$\int_{\mathfrak{h}}^{\infty} \sum_{\zeta=1}^{2} \mathfrak{X}\left(\tau_\zeta(t)\right) dt = \int_{\mathfrak{h}}^{\infty} [\mathfrak{X}\left(\tau_\zeta(t)\right) + \mathfrak{X}\left(\tau_\zeta(t)\right)]\, dt = \int_{\mathfrak{h}}^{\infty} \frac{30}{(t+1)^4} dt = \frac{10}{(\mathfrak{h}+1)^3} \tag{2.18}$$

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https://publishoa.com
ISSN: 1309-3452

$$\frac{\mathfrak{p}_1\mathfrak{q}_1}{\mathfrak{p}_2\mu_2}\int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{2}\mathfrak{L}\left(\tau_\zeta(t)\right)dr\,dt\,ds+\frac{1}{\mathfrak{p}_2\mu_2}\mathfrak{X}(\mathrm{t})=\frac{2}{\mathfrak{h}+1}+\frac{15}{4\mathfrak{h}^4} \tag{2.19}$$

from eq. (2.18) and (2.19), we have

$$\frac{10}{(\mathfrak{h}+1)^3}\preccurlyeq\frac{2}{\mathfrak{h}+1}+\frac{15}{4\mathfrak{h}^2},\mathfrak{h}\succcurlyeq 1$$

Thus

$$\int_{\mathfrak{h}}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathfrak{X}\left(\tau_\varsigma(\mathfrak{h})\right)dt\preccurlyeq\frac{1}{\mathfrak{p}_2\mu_2}\left(\mathfrak{p}_1\mathfrak{q}_1\int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathfrak{L}\left(\tau_\varsigma(t)\right)dr\,dt\,ds+\mathfrak{X}(\mathrm{t})\right)\preccurlyeq\mathcal{M}$$

Now,

$$\frac{\mathfrak{p}_2\mathfrak{q}_2}{\mathfrak{p}_1\mu_1}\int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{2}\mathfrak{X}\left(\tau_\varsigma(t)\right)dr\,dt\,ds+\frac{1}{\mathfrak{p}_1\mu_1}\mathfrak{L}(\mathrm{t})=\frac{20}{\mathfrak{h}+1}+\frac{12}{\mathfrak{h}^4} \tag{2.20}$$

$$\int_{\mathfrak{h}}^{\infty}\sum_{\zeta=1}^{2}\mathfrak{L}\left(\tau_\zeta(t)\right)dt=\frac{8}{(\mathfrak{h}+1)^3} \tag{2.21}$$

$$\frac{1}{\mathfrak{p}_1\mu_1}\left(\mathfrak{p}_2\mathfrak{q}_2\int_{\mathfrak{h}}^{\infty}\int_{s}^{\infty}\int_{r}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathfrak{X}\left(\tau_\zeta(t)\right)dr\,dt\,ds+\mathfrak{L}(\mathrm{t})\right)\preccurlyeq\int_{\mathfrak{h}}^{\infty}\sum_{\zeta=1}^{\Gamma}\mathfrak{L}\left(\tau_\zeta(t)\right)dt,\qquad \mathfrak{h}\succcurlyeq\mathfrak{h}_1$$

3. Property of oscillation for third order multiple delay differential equation:

In the present section, we'll seek for oscillatory criteria to Eq. (1.1) and we use some basic lemmas:

Lemma 3.1 [18]:

Let $\mathcal{Z}\in C^{\Gamma}[\mathfrak{R},\mathfrak{R}]$ and $\mathcal{Z}^{(\Gamma)}(\mathfrak{h})\mathcal{Z}^{(\Gamma-1)}(\mathfrak{h})>0,\ \mathfrak{h}\geq\mathfrak{h}_0,\mathfrak{h}\in(-\infty,\infty)$

Then the following statements hold

1. If $\mathcal{Z}^{(\Gamma)}(\mathfrak{h})$is positive for $\mathfrak{h}\geq\mathfrak{h}_0$ then $\mathcal{Z}^{(\zeta)}(\mathfrak{h})$ is increasing for $\mathfrak{h}\geq\mathfrak{h}_0$ and $\lim\limits_{\mathfrak{h}\to\infty}\mathcal{Z}^{(\zeta)}(\mathfrak{h})=\infty$ for $\zeta=\Gamma-1,\Gamma-2,\dots,0$
2. If $\mathcal{Z}^{(\Gamma)}(\mathfrak{h})$is negative for $\mathfrak{h}\geq\mathfrak{h}_0$ then $\mathcal{Z}^{(\zeta)}(\mathfrak{h})$ is decreasing for $\mathfrak{h}\geq\mathfrak{h}_0$ and $\lim\limits_{\mathfrak{h}\to\infty}\mathcal{Z}^{(\zeta)}(\mathfrak{h})=-\infty$ for $\zeta=\Gamma-1,\Gamma-2,\dots,0$

Then $\mathcal{Z}(\mathfrak{h})$ cannot be negative for $\mathfrak{h}\succcurlyeq\mathfrak{h}_1\succcurlyeq\mathfrak{h}_0$.

Lemma 3.2 [19]:

Assume that $\varepsilon,\varrho\in C[\mathfrak{R}^+,\mathfrak{R}^+]$ are continuous functions such that $\varepsilon(\mathfrak{h})\prec\mathfrak{h},\varepsilon'(\mathfrak{h})\succcurlyeq 0$ for $\mathfrak{h}\succcurlyeq\mathfrak{h}_0$ with $\lim\limits_{\mathfrak{h}\to\infty}\varepsilon(\mathfrak{h})=\infty$.

If

$$\liminf_{\mathfrak{h}\to\infty}\int_{\varepsilon(\mathfrak{h})}^{\mathfrak{h}}\varrho(s)\,ds\succ\frac{1}{e} \tag{3.1}$$

then the inequality

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https://publishoa.com
ISSN: 1309-3452

$$\mathcal{Z}'(\mathfrak{h}) + \varrho(\mathfrak{h})\mathcal{Z}(\varepsilon(\mathfrak{h})) \preccurlyeq 0$$

has no eventually positive solution.

Lemma 3.3: Assume that:

$$\mathcal{J}(\mathfrak{h}) = \mathcal{Z}(\mathfrak{h}) - \sum_{\zeta=1}^{\Gamma} \int_{T}^{\mathfrak{h}} \int_{T}^{\mathfrak{r}} \int_{s}^{\tau_\zeta^{-1}(\varepsilon_\zeta(s))} \mathcal{A}_\zeta(\mathrm{t})\mathcal{T}_\zeta\left(\mathcal{Z}\left(\tau_\zeta(\mathrm{t})\right)\right) d\mathrm{t}\, ds\, d\mathfrak{r}$$

$$-\sum_{\zeta=1}^{\Gamma} \mathcal{B}_\zeta(\mathfrak{h})\mathcal{S}_\zeta\left(\mathfrak{h}, \mathcal{Z}\left(\tau_\zeta(\mathfrak{h})\right)\right) \qquad (3.2)$$

And the following assumptions hold:

$$\text{H1:}\, \vartheta_2(\mathfrak{h}) \le \frac{\mathcal{S}_\zeta\left(\mathfrak{h}, \mathcal{Z}\left(\tau_\zeta(\mathfrak{h})\right)\right)}{\mathcal{Z}\left(\tau_\zeta(\mathfrak{h})\right)} \le \frac{\mathcal{T}_\zeta\left(\mathcal{Z}\left(\tau_\zeta(\mathfrak{h})\right)\right)}{\mathcal{Z}\left(\tau_\zeta(\mathfrak{h})\right)} \le \vartheta_1(\mathfrak{h}), \qquad \rho(\mathfrak{h}) = \max\{\tau_\zeta(\mathfrak{h})\}$$

$$\text{H2:}\, \lim_{\mathfrak{h}\to\infty} \inf \sum_{\zeta=1}^{\Gamma} \left[\int_{T}^{\alpha_\upsilon} \int_{T}^{\mathfrak{r}} \int_{s}^{\tau_\zeta^{-1}(\varepsilon_\zeta(s))} \mathcal{A}_\zeta(\mathrm{t})\vartheta_1(\mathrm{t}) d\mathrm{t}\, ds\, d\mathfrak{r} + \sum_{\zeta=1}^{\Gamma} \mathcal{B}_\zeta(\alpha_\upsilon)\vartheta_1(\mathfrak{h}) \right] \le 1$$

If $\mathcal{Z}(\mathfrak{h})$ is eventually positive bounded solution of Eq. (1.1) with $(\tau_\zeta^{-1}\left(\varepsilon_\zeta(\mathfrak{h})\right))' \ge 0$ then:

$\mathcal{J}(\mathfrak{h})$ positive non-increasing function.

Proof. Assume that a solution $\mathcal{Z}(\mathfrak{h})$ is a non-oscillatory bounded solution of the Eq.(1.1). So, suppose that $\mathcal{Z}(\mathfrak{h})$ is eventually positive bounded solution, there is $\mathfrak{h}_1 \ge \mathfrak{h}_0 + \rho$ such that $\mathcal{Z}(\mathfrak{h}) > 0$ for $\mathfrak{h} \ge \mathfrak{h}_1$.

$$\frac{d}{d\mathfrak{h}}\mathcal{J}(\mathfrak{h}) = \frac{d}{d\mathfrak{h}}\mathcal{Z}(\mathfrak{h}) - \sum_{\zeta=1}^{\Gamma} \int_{T}^{\mathfrak{h}} \int_{s}^{\tau_\zeta^{-1}(\varepsilon_\zeta(s))} \mathcal{A}_\zeta(\mathrm{t})\mathcal{T}_\zeta\left(\mathcal{Z}\left(\tau_\zeta(\mathrm{t})\right)\right) d\mathrm{t}\, ds - \frac{d}{d\mathfrak{h}}\sum_{\zeta=1}^{\Gamma} \mathcal{B}_\zeta(\mathfrak{h})\mathcal{S}_\zeta\left(\mathfrak{h}, \mathcal{Z}\left(\tau_\zeta(\mathfrak{h})\right)\right)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\mathfrak{h}^2}\mathcal{J}(\mathfrak{h}) = \frac{\mathrm{d}^2}{\mathrm{d}\mathfrak{h}^2}\,\mathcal{Z}(\mathfrak{h}) - \sum_{\zeta=1}^{\Gamma} \int_{\mathfrak{h}}^{\tau_\zeta^{-1}(\varepsilon_\zeta(\mathfrak{h}))} \mathcal{A}_\zeta(\mathrm{t})\mathcal{T}_\zeta\left(\mathcal{Z}\left(\tau_\zeta(\mathrm{t})\right)\right) d\mathrm{t} - \frac{\mathrm{d}^2}{\mathrm{d}\mathfrak{h}^2}\sum_{\zeta=1}^{\Gamma} \mathcal{B}_\zeta(\mathfrak{h})\mathcal{S}_\zeta\left(\mathfrak{h}, \mathcal{Z}\left(\tau_\zeta(\mathfrak{h})\right)\right)$$

$$\begin{aligned}\frac{\mathrm{d}^3}{\mathrm{d}\mathfrak{h}^3}\mathcal{J}(\mathfrak{h}) = \frac{\mathrm{d}^3}{\mathrm{d}\mathfrak{h}^3}\,\mathcal{Z}(\mathfrak{h}) & \\ -\sum_{\zeta=1}^{\Gamma} \Bigg[& \mathcal{A}_\zeta\left(\tau_\zeta^{-1}\left(\varepsilon_\zeta(\mathfrak{h})\right)\right)\mathcal{T}_\zeta\left(\mathcal{Z}\left(\tau_\zeta\left(\tau_\zeta^{-1}\left(\varepsilon_\zeta(\mathfrak{h})\right)\right)\right)\right)(\tau_\zeta^{-1}\left(\varepsilon_\zeta(\mathfrak{h})\right))' \\ & - \mathcal{A}_\zeta(\mathfrak{h})\mathcal{T}_\zeta\left(\mathcal{Z}\left(\tau_\zeta(\mathfrak{h})\right)\right)\Bigg] - \frac{\mathrm{d}^3}{\mathrm{d}\mathfrak{h}^3}\sum_{\zeta=1}^{\Gamma} \mathcal{B}_\zeta(\mathfrak{h})\mathcal{S}_\zeta\left(\mathfrak{h}, \mathcal{Z}\left(\tau_\zeta(\mathfrak{h})\right)\right)\end{aligned}$$

From equation (1.1), we obtain that:

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https://publishoa.com
ISSN: 1309-3452

$$\frac{\mathrm{d}^3}{\mathrm{d}\mathfrak{h}^3}\mathcal{J}(\mathfrak{h}) = -\sum_{\zeta=1}^{\Gamma} \mathcal{A}_\zeta(\mathfrak{h})\mathcal{T}_\zeta\left(\mathcal{Z}\left(\tau_\zeta(\mathfrak{h})\right)\right) + \frac{\mathrm{d}^3}{\mathrm{d}\mathfrak{h}^3}\sum_{\zeta=1}^{\Gamma} \mathcal{B}_\zeta(\mathfrak{h})\mathcal{S}_\zeta\left(\mathfrak{h}, \mathcal{Z}\left(\tau_\zeta(\mathfrak{h})\right)\right)$$
$$-\sum_{\zeta=1}^{\Gamma}\left[\mathcal{A}_\zeta\left(\tau_\zeta^{-1}\left(\varepsilon_\zeta(\mathfrak{h})\right)\right)\mathcal{T}_\zeta\left(\mathcal{Z}\left(\varepsilon_\zeta(\mathfrak{h})\right)\right)(\tau_\zeta^{-1}\left(\varepsilon_\zeta(\mathfrak{h})\right))' - \mathcal{A}_\zeta(\mathfrak{h})\mathcal{T}_\zeta\left(\mathcal{Z}\left(\tau_\zeta(\mathfrak{h})\right)\right)\right]$$
$$-\frac{\mathrm{d}^3}{\mathrm{d}\mathfrak{h}^3}\sum_{\zeta=1}^{\Gamma} \mathcal{B}_\zeta(\mathfrak{h})\mathcal{S}_\zeta\left(\mathfrak{h}, \mathcal{Z}\left(\tau_\zeta(\mathfrak{h})\right)\right)$$

$$\frac{\mathrm{d}^3}{\mathrm{d}\mathfrak{h}^3}\mathcal{J}(\mathfrak{h}) = -\sum_{\zeta=1}^{\Gamma} \mathcal{A}_\zeta\left(\tau_\zeta^{-1}\left(\varepsilon_\zeta(\mathfrak{h})\right)\right)\mathcal{T}_\zeta\left(\mathcal{Z}\left(\varepsilon_\zeta(\mathfrak{h})\right)\right)(\tau_\zeta^{-1}\left(\varepsilon_\zeta(\mathfrak{h})\right))' \leq 0 \qquad (3.5)$$

So, we conclude that $\frac{\mathrm{d}^3}{\mathrm{d}\mathfrak{h}^3}\mathcal{J}(\mathfrak{h}) \leq 0$ and $\frac{\mathrm{d}^2}{\mathrm{d}\mathfrak{h}^2}\mathcal{J}(\mathfrak{h}), \frac{d}{d\mathfrak{h}}\mathcal{J}(\mathfrak{h}), \mathcal{J}(\mathfrak{h})$ are monotone (nonincreasing) functions. We have two cases to consider:

Case1:

$$\text{If } \frac{\mathrm{d}^3}{\mathrm{d}\mathfrak{h}^3}\mathcal{J}(\mathfrak{h}) \leq 0 \text{ with } \frac{\mathrm{d}^2}{\mathrm{d}\mathfrak{h}^2}\mathcal{J}(\mathfrak{h}) \leq 0, \frac{d}{d\mathfrak{h}}\mathcal{J}(\mathfrak{h}) \leq 0 \text{ and } \mathcal{J}(\mathfrak{h}) \leq 0$$

for $\mathfrak{h} \geq \mathfrak{h}_1$ by Lemma 3.1 it follows that $\lim_{\mathfrak{h}\to\infty}\mathcal{J}(\mathfrak{h}) = -\infty$ and with (3.4) we imply that $\lim_{\mathfrak{h}\to\infty}\mathcal{Z}(\mathfrak{h}) = -\infty$, which is a contradiction.

Case 2:

$$\text{If } \frac{\mathrm{d}^3}{\mathrm{d}\mathfrak{h}^3}\mathcal{J}(\mathfrak{h}) \leq 0 \text{ and } \frac{\mathrm{d}^2}{\mathrm{d}\mathfrak{h}^2}\mathcal{J}(\mathfrak{h}) \geq 0, \text{ we claim that } \frac{d}{d\mathfrak{h}}\mathcal{J}(\mathfrak{h}) \leq 0, \ \mathfrak{h} \geq \mathfrak{h}_1$$

Otherwise $\frac{d}{d\mathfrak{h}}\mathcal{J}(\mathfrak{h}) \geq 0$, by Lemma 3.1 it follows that $\lim_{\xi\to\infty}\mathcal{J}(\mathfrak{h}) = \infty$ and with (3.4) we imply that $\lim_{\mathfrak{h}\to\infty}\mathcal{Z}(\mathfrak{h}) = \infty$, which is a contradiction.

$$\text{So } \frac{d}{d\mathfrak{h}}\mathcal{J}(\mathfrak{h}) \leq 0, \ \mathfrak{h} \geq \mathfrak{h}_1,$$

we claim that $\mathcal{J}(\mathfrak{h}) \geq 0, \ \mathfrak{h} \geq \mathfrak{h}_1$

Otherwise $\mathcal{J}(\mathfrak{h}) \leq 0$, so there exist $\varphi < 0$ such that $\mathcal{J}(\mathfrak{h}) \leq \varphi, \ \mathfrak{h} \geq \mathfrak{h}_2 \geq \mathfrak{h}_1$

Then from (3.2):

$$\mathcal{Z}(\mathfrak{h}) \leq \varphi + \sum_{\zeta=1}^{\Gamma}\int_T^{\mathfrak{h}}\int_T^{\mathfrak{r}}\int_s^{\tau_\zeta^{-1}\left(\varepsilon_\zeta(s)\right)} \mathcal{A}_\zeta(\mathrm{t})\mathcal{T}_\zeta\left(\mathcal{Z}\left(\tau_\zeta(\mathrm{t})\right)\right)dt\, ds\, d\mathfrak{r} + \sum_{\zeta=1}^{\Gamma}\mathcal{B}_\zeta(\mathfrak{h})\mathcal{S}_\zeta\left(\mathfrak{h}, \mathcal{Z}\left(\tau_\zeta(\mathfrak{h})\right)\right)$$

Since $\mathcal{Z}(\mathfrak{h})$ is bounded then $\lim_{\mathfrak{h}\to\infty}\inf\mathcal{Z}(\mathfrak{h}) = \psi, \ 0 \leq \psi < \infty$

So there is a sequence $\{\alpha_{\boldsymbol{\upsilon}}\}$, such that $\lim_{\boldsymbol{\upsilon}\to\infty}\alpha_{\boldsymbol{\upsilon}} = \infty$ and $\lim_{\boldsymbol{\upsilon}\to\infty}\mathcal{Z}(\alpha_{\boldsymbol{\upsilon}}) = \varphi$

$$\boldsymbol{\varrho}_1(\mathfrak{h}) = \min\{\tau_\varsigma(\mathfrak{h})\} \text{ and } \boldsymbol{\varrho}_2(\mathfrak{h}) = \max\{\tau_\varsigma(\mathfrak{h})\}, \mathfrak{h} \geq \mathfrak{h}_2$$

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ISSN: 1309-3452

$$\mathcal{Z}(\gamma_{\boldsymbol{\upsilon}}) = \max\{\mathcal{Z}(\mathfrak{h}), \boldsymbol{\varrho}_1(\alpha_{\boldsymbol{\upsilon}}) \le \mathfrak{h} \le \boldsymbol{\varrho}_2(\alpha_{\boldsymbol{\upsilon}})\}$$

$$\text{So } \mathcal{Z}(\gamma_{\boldsymbol{\upsilon}}) \ge \mathcal{Z}(\tau_\zeta(\mathfrak{h}))$$

$$\lim_{\boldsymbol{\upsilon}\to\infty} \gamma_{\boldsymbol{\upsilon}} = \infty \text{ and } \lim_{\boldsymbol{\upsilon}\to\infty} \inf \mathcal{Z}(\gamma_{\boldsymbol{\upsilon}}) \ge \psi$$

$$\mathcal{Z}(\alpha_{\boldsymbol{\upsilon}}) \le \varphi + \sum_{\zeta=1}^{\Gamma} \int_T^{\alpha_{\boldsymbol{\upsilon}}} \int_T^{\mathfrak{r}} \int_s^{\tau_\zeta^{-1}\left(\varepsilon_\zeta(s)\right)} \mathcal{A}_\zeta(\mathrm{t}) \mathcal{T}_\zeta\left(\mathcal{Z}\left(\tau_\zeta(\mathrm{t})\right)\right) dt\, ds\, d\mathfrak{r} + \sum_{\zeta=1}^{\Gamma} \mathcal{B}_\zeta(\alpha_{\boldsymbol{\upsilon}}) \mathcal{S}_\zeta\left(\mathfrak{h}, \mathcal{Z}\left(\tau_\zeta(\alpha_{\boldsymbol{\upsilon}})\right)\right)$$

$$\mathcal{Z}(\alpha_{\boldsymbol{\upsilon}}) \le \varphi + \sum_{\zeta=1}^{\Gamma} \int_T^{\alpha_{\boldsymbol{\upsilon}}} \int_T^{\mathfrak{r}} \int_s^{\tau_\zeta^{-1}\left(\varepsilon_\zeta(s)\right)} \mathcal{A}_\zeta(\mathrm{t}) \vartheta_1(\mathrm{t}) \mathcal{Z}\left(\tau_\zeta(\mathrm{t})\right) d\mathrm{t}\, ds\, d\mathfrak{r} + \sum_{\zeta=1}^{\Gamma} \mathcal{B}_\zeta(\alpha_{\boldsymbol{\upsilon}}) \vartheta_1(\alpha_{\boldsymbol{\upsilon}}) \mathcal{Z}\left(\tau_\zeta(\alpha_{\boldsymbol{\upsilon}})\right)$$

$$\mathcal{Z}(\alpha_{\boldsymbol{\upsilon}}) \le \varphi + \sum_{\zeta=1}^{\Gamma} \mathcal{Z}(\gamma_{\boldsymbol{\upsilon}}) \left\{ \int_T^{\alpha_{\boldsymbol{\upsilon}}} \int_T^{\mathfrak{r}} \int_s^{\tau_\zeta^{-1}\left(\varepsilon_\zeta(s)\right)} \mathcal{A}_\zeta(\mathrm{t}) \vartheta_1(\mathrm{t}) d\mathrm{t}\, ds\, d\mathfrak{r} + \sum_{\zeta=1}^{\Gamma} \mathcal{B}_\zeta(\alpha_{\boldsymbol{\upsilon}}) \vartheta_1(\alpha_{\boldsymbol{\upsilon}}) \right\}$$

By taking limit inferior to the both sides of the last inequality as $\boldsymbol{\upsilon} \to \infty$, it follows that:
$\psi \le \varphi + \psi$ which is a contradiction.

Theorem 3.1

Assume that all conditions of Lemma 3.3 hold and $\mathcal{J}(\mathfrak{h})$ is defined as in (3.2) with $\varepsilon_\zeta(\mathfrak{h}) < \mathfrak{h}, \tau_\zeta(\mathfrak{h}) < \mathfrak{h}$ and $(\tau_\zeta^{-1}(\varepsilon_\zeta(\mathfrak{h})))' = -\eta(\mathfrak{h})$ in addition to the condition:

$$\lim_{\mathfrak{h}\to\infty} \inf \sum_{\zeta=1}^{\Gamma} \left[\left\{ \int_{\varepsilon_\zeta(\delta(\mathfrak{h}))}^{\mathfrak{h}} \int_r^{\delta(r)} \int_T^{s} \mathcal{A}_\zeta\left(\tau_\zeta^{-1}\left(\varepsilon_\zeta(\mathrm{t})\right)\right) \vartheta_2(\tau_\zeta^{-1}(\varepsilon_\zeta(\mathrm{t}))) \eta(\mathfrak{h}) d\mathrm{t}\, ds\, dr \right\} \right] \ge \frac{1}{e} \qquad (3.4)$$

Then every solution of Eq. (1.1) oscillates.

Proof

Assume that a solution $\mathcal{Z}(\mathfrak{h})$ is a non-oscillatory of the Eq. (1.1). So, let $\mathcal{Z}(\mathfrak{h})$ is eventually positive solution, there is $\mathfrak{h}_1 \geqslant \mathfrak{h}_0 + \boldsymbol{\varrho}_2(\mathfrak{h}), \ni \mathcal{Z}(\mathfrak{h}) \succ 0, \ \mathfrak{h} \geqslant \mathfrak{h}_1$.
Integrating (3.3) from T to $\mathfrak{h}$, $0 \le T \le \mathfrak{h}$:

$$\frac{\mathrm{d}^2}{\mathrm{d}\mathfrak{h}^2} \mathcal{J}(\mathfrak{h}) - \frac{\mathrm{d}^2}{\mathrm{d}\mathfrak{h}^2} \mathcal{J}(T) = -\sum_{\zeta=1}^{\Gamma} \int_T^{\mathfrak{h}} \mathcal{A}_\zeta\left(\tau_\zeta^{-1}\left(\varepsilon_\zeta(\mathrm{t})\right)\right) \mathcal{T}_\zeta\left(\mathcal{Z}\left(\varepsilon_\zeta(\mathrm{t})\right)\right) (\tau_\zeta^{-1}\left(\varepsilon_\zeta(\mathrm{t})\right))' dt$$

Integrating the last equation from $\mathfrak{h}$ to $\delta(\mathfrak{h}), \delta(\mathfrak{h}) > \mathfrak{h}, \varepsilon_\zeta(\delta(\mathfrak{h})) < \mathfrak{h}, \lim_{\mathfrak{h}\to\infty} \varepsilon_\zeta(\delta(\mathfrak{h})) = \infty, \zeta = 1,2,\dots,\Gamma$

$$\frac{\mathrm{d}}{\mathrm{d}\mathfrak{h}} \mathcal{J}(\delta(\mathfrak{h})) - \frac{\mathrm{d}}{\mathrm{d}\mathfrak{h}} \mathcal{J}(\mathfrak{h}) = \sum_{\zeta=1}^{\Gamma} \int_{\mathfrak{h}}^{\delta(\mathfrak{h})} \int_T^{s} \mathcal{A}_\zeta\left(\tau_\zeta^{-1}\left(\varepsilon_\zeta(\mathrm{t})\right)\right) \mathcal{T}_\zeta\left(\mathcal{Z}\left(\varepsilon_\zeta(\mathrm{t})\right)\right) \eta(\mathfrak{h}) d\mathrm{t}\, ds$$

$$\frac{\mathrm{d}}{\mathrm{d}\mathfrak{h}} \mathcal{J}(\delta(\mathfrak{h})) - \frac{\mathrm{d}}{\mathrm{d}\mathfrak{h}} \mathcal{J}(\mathfrak{h}) \ge \sum_{\zeta=1}^{\Gamma} \int_{\mathfrak{h}}^{\delta(\mathfrak{h})} \int_T^{s} \mathcal{A}_\zeta\left(\tau_\zeta^{-1}\left(\varepsilon_\zeta(\mathrm{t})\right)\right) \vartheta_2(\tau_\zeta^{-1}(\varepsilon_\zeta(\mathrm{t}))) \mathcal{Z}\left(\varepsilon_\zeta(\mathrm{t})\right) \eta(\mathfrak{h}) dt\, ds$$

But from (3.2) $\mathcal{Z}(\mathfrak{h}) \ge \mathcal{J}(\mathfrak{h})$:

$$\frac{\mathrm{d}}{\mathrm{d}\mathfrak{h}}\mathcal{J}(\delta(\mathfrak{h}))-\frac{\mathrm{d}}{\mathrm{d}\mathfrak{h}}\mathcal{J}(\mathfrak{h})\geq\sum_{\zeta=1}^{\Gamma}\int_{\mathfrak{h}}^{\delta(\mathfrak{h})}\int_{T}^{s}\mathcal{A}_{\zeta}\left(\tau_{\zeta}^{-1}\left(\varepsilon_{\zeta}(\mathrm{t})\right)\right)\vartheta_{2}\left(\tau_{\varsigma}^{-1}\left(\varepsilon_{\zeta}(\mathrm{t})\right)\right)\mathcal{J}\left(\varepsilon_{\zeta}(\mathrm{t})\right)\eta(\mathfrak{h})dt\,ds$$

$$\frac{\mathrm{d}}{\mathrm{d}\mathfrak{h}}\mathcal{J}(\delta(\mathfrak{h}))-\frac{\mathrm{d}}{\mathrm{d}\mathfrak{h}}\mathcal{J}(\mathfrak{h})\geq\sum_{\zeta=1}^{\Gamma}\mathcal{J}\left(\varepsilon_{\zeta}(\delta(\mathfrak{h}))\right)\int_{\mathfrak{h}}^{\delta(\mathfrak{h})}\int_{T}^{s}\mathcal{A}_{\zeta}\left(\tau_{\zeta}^{-1}\left(\varepsilon_{\zeta}(\mathrm{t})\right)\right)\vartheta_{2}\left(\tau_{\varsigma}^{-1}\left(\varepsilon_{\zeta}(\mathrm{t})\right)\right)\eta(\mathfrak{h})dt\,ds$$

$$-\frac{\mathrm{d}}{\mathrm{d}\mathfrak{h}}\mathcal{J}(\mathfrak{h})\geq\sum_{\zeta=1}^{\Gamma}\mathcal{J}\left(\varepsilon_{\zeta}(\delta(\mathfrak{h}))\right)\int_{\mathfrak{h}}^{\delta(\mathfrak{h})}\int_{T}^{s}\mathcal{A}_{\zeta}\left(\tau_{\zeta}^{-1}\left(\varepsilon_{\zeta}(\mathrm{t})\right)\right)\vartheta_{2}\left(\tau_{\varsigma}^{-1}\left(\varepsilon_{\zeta}(\mathrm{t})\right)\right)\eta(\mathfrak{h})dt\,ds$$

$$\frac{\mathrm{d}}{\mathrm{d}\mathfrak{h}}\mathcal{J}(\mathfrak{h})+\sum_{\zeta=1}^{\Gamma}\mathcal{J}\left(\varepsilon_{\zeta}(\delta(\mathfrak{h}))\right)\int_{\mathfrak{h}}^{\delta(\mathfrak{h})}\int_{T}^{s}\mathcal{A}_{\zeta}\left(\tau_{\zeta}^{-1}\left(\varepsilon_{\zeta}(\mathrm{t})\right)\right)\vartheta_{2}\left(\tau_{\varsigma}^{-1}\left(\varepsilon_{\zeta}(\mathrm{t})\right)\right)\eta(\mathfrak{h})dt\,ds\leq 0$$

By Lemma 3.2 then the last inequality has no eventually positive solution.

4. Conclusions

We conclude that the novel conditions to demonstrate the existence of no oscillatory bounded solution to the differential equation of kind (TOMDDE) were very efficient and reliable. The illustrative example explained the quickness of calculations. Furthermore, the new conditions for lemma 3.3 were harmonious with theorem 3.1 to acquire the sufficient conditions for oscillatory solution. are more flexible and easy to apply in examples.

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