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On a New Class of Analytic Univalent Tasks Specified Through Derivative of Ruscheweyh Operator

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Abstract

In the current work, we introduce a class as new $M(\delta,\beta,\gamma,\lambda)$ of analytic univalent tasks specified through derivative of Ruscheweyh operative.

For the tasks belonging to this class, we obtain Coefficient inequalities, closure theorem, theorem growth and distortion, star likeness radii and convexity, arithmetic mean, linear combination, and Hadamard product. 2000 Grouping of mathematics subjects: 30C45.

Keywords and Phrases: derivative of Ruscheweyh, growth and distortion, arithmetic mean, linear combination, Hadamard product.

INTRODUCTION

Assume R signify the fotm tasks class:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

That is univalent and analytic in $U = \{z: |z| < 1\}$.

If a function f is offered through (1) and g is specified through

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$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{2}$$

Is in the R class, at that moment the ($f$and g) Hadamard product or convolution is specified through:

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \qquad z \in U \tag{3}$$

Assume M denoted the R sub class having form tasks:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad a_k \geq 0 \tag{4}$$

A function $f \in R$ is said to be "univalent starlike of order α $(0 \leq \alpha < 1)$ in U if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \tag{5}$$

Similarly, a function $f(z)$ is univalent of order convex α $(0 \leq \alpha < 1)$ in U if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \tag{6}$$

We aim to study to subclass $M(\delta, \beta, \gamma, \lambda)$ having tasks $f \in M$ and adequate:

$$\left|\frac{\delta z\left(D^\lambda f(z)\right)''}{\beta[z(D^\lambda f(z))'' + (D^\lambda f(z))'] + (1-\delta)(D^\lambda f(z))'}\right| < \gamma. \tag{7}$$

$$0 \leq \delta < 1 \;, 0 \leq \beta < 1, \qquad 0 < \gamma < 1, \;\; z \in U.$$

And $D^\lambda f(z)$ is specified as following

$$D^\lambda f(z) = z + \sum_{k=2}^{\infty} a_k B_k(\lambda) z^k,$$

Where

$$B_k(\lambda) = \frac{(\lambda+1)(\lambda+2)\Lambda(\lambda+k-1)}{(k-1)!}, \lambda > -1, z \in U \tag{8}$$

This function is named the derivative of Ruscheweyh [5],[6], of f of order λ denoted through $D^\lambda f$.

Another studied classes through SERAP BULUT[4], Atshan and Mustafa and

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Mouajeeb [3] and Xiao-FeiLi and An-ping Wang[7].

In [1], [2], for other classes, few of the following characters are deliberated

2. Inequality as Coefficient:

In the theorem as follow, for function to be in the class$M(\delta,\beta,\gamma,\lambda)$, we get an essential and adequate condition:

Theorem 1: Assume the function f be specified through (4) at that time $f \in M(\delta,\beta,\gamma,\lambda)$ if and single if

$$\sum_{k=2}^{\infty} k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)\, a_k \;\le\; \gamma(\beta+1-\delta)\,, \quad (9)$$

where

$$0 \le \delta < 1\ ,0 \le \beta < 1\ ,0 < \gamma < 1, \lambda > -1.$$

For the function f, the finding is sharp offered through

$$f(z) = z + \frac{\gamma(\beta+1-\delta)}{k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)} z^k, \qquad k \ge 2\,. \quad (10)$$

Evidence: Expect that the inequality (9) as of true and$|z| = 1.$, At that time from (7) we obtain

$$\left|\delta z\big(D^\lambda f(z)\big)''\right| - \gamma\left|\beta\left[z\big(D^\lambda f(z)\big)'' + \big(D^\lambda f(z)\big)'\right] + (1-\delta)\big(D^\lambda f(z)\big)'\right|$$

$$= \left|\delta \sum_{k=2}^{\infty} k(k-1)B_k(\lambda)\, a_k z^{k-1}\right| - \gamma\left|\beta + (1-\delta) + \sum_{k=2}^{\infty} k[\beta(k-1)+\beta+1-\delta]B_k(\lambda)\, a_k z^{k-1}\right|$$

$$\le \sum_{k=2}^{\infty} k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)\, a_k - \gamma\,(\beta+1-\delta) \le 0.$$

Hence, through maximum modulus principle, $f \in M(\delta,\beta,\gamma,\lambda)$.

On the contrary, expect that$f \in M(\delta,\beta,\gamma,\lambda)$. At that time from (7), we get

$$\left|\frac{\delta z\big(D^\lambda f(z)\big)''}{\beta[z(D^\lambda f(z))'' + (D^\lambda f(z))'] + (1-\delta)(D^\lambda f(z))'}\right|$$

$$= \left|\frac{\delta \sum_{k=2}^{\infty} k(k-1)B_k(\lambda)\, a_k z^{k-1}}{\beta + (1-\delta) + \sum_{k=2}^{\infty} k[\beta(k-1)+\beta+1-\delta]B_k(\lambda)\, a_k z^{k-1}}\right| < \gamma.$$

Since $Re(z) \le |z|$ for whole z $(z \in U)$, we acquire

$$Re\left\{\frac{\delta \sum_{k=2}^{\infty} k(k-1)B_k(\lambda)\, a_k z^{k-1}}{\beta + (1-\delta) + \sum_{k=2}^{\infty} k[\beta(k-1)+\beta+1-\delta]B_k(\lambda)\, a_k z^{k-1}}\right\} < \gamma. \quad (11)$$

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We are able to choose the z value on the actual axis hence $(D^{\lambda} f(z))''$ is actual. Assume $z \to 1^-$ via actual values, thus we able to state (11) as

$$\sum_{k=2}^{\infty} k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)\, a_k \le \gamma(\beta+1-\delta).$$

And such completes the evidence.

Corollary 1: Assume thr function $f \in M(\delta,\beta,\gamma,\lambda)$. At that time

$$a_k \le \frac{\gamma(\beta+1-\delta)}{k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)} \ , k \ge 2,$$

3. Theorem as closure:

Theorem 2: Assume the tasks f_r specified through

$$f_s(z) = z + \sum_{k=2}^{\infty} a_{k,r}\, z^k \ , \left(a_{k,r} \ge 0, \quad r = 1,2,\dots,l\right),$$

As it is in the $M(\delta,\beta,\gamma,\lambda)$ class for each $r = 1,2,\dots,l$ .At that time the function h specified through

$$h(z) = z + \sum_{k=2}^{\infty} e_k\, z^k, \ \ (e_k \ge 0\,),$$

also belongs to the $M(\delta,\beta,\gamma,\lambda)$ class, in which

$$e_k = \frac{1}{l}\sum_{r=1}^{l} a_{k,r} \quad , \quad (k \ge 2).$$

Evidence: Since $f_r \in M(\delta,\beta,\gamma,\lambda)$, at that time through Theorem 1, we obtain

$$\sum_{k=2}^{\infty} k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)a_{k,r} \le \gamma(\beta+1-\delta)\,, \qquad (12)$$

For each $r = 1,2,\dots,l$. Hence

$$\sum_{k=2}^{\infty} k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)e_k$$

$$= \sum_{k=2}^{\infty} k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)\left(\frac{1}{l}\sum_{r=1}^{l} a_{k,r}\right)$$

$$= \frac{1}{l}\sum_{r=1}^{l}\left(\sum_{k=2}^{\infty} k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)\right) \le \gamma(\beta+1-\delta).$$

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Through Theorem1, it is following $h \in M(\delta,\beta,\gamma,\lambda)$.

4. Growth and Distortion Theorems:

Theorem 3: If$f \in M(\delta,\beta,\gamma,\lambda)$, at that time

$$|z|-\frac{\gamma(\beta+1-\delta)}{2[\delta+\gamma(2\beta+1-\delta)](1+\lambda)}|z|^2 \leq |f(z)| \leq |z|+\frac{\gamma(\beta+1-\delta)}{2[\delta+\gamma(2\beta+1-\delta)](1+\lambda)}|z|^2 . \quad (13)$$

The finding is sharp for the $f(z)$ function offered through:

$$f(z)=z+\frac{\gamma(\beta+1-\delta)}{2[\delta+\gamma(2\beta+1-\delta)](1+\lambda)}z^2 \quad .$$

Evidence:

$$|f(z)|=\left|z+\sum_{k=2}^{\infty}a_k z^k\right| \leq |z|+\sum_{k=2}^{\infty}a_k|z|^k$$

$$\leq |z|+|z|^2\sum_{k=2}^{\infty}a_k .$$

ThroughTheorem1

$$\sum_{k=2}^{\infty}a_k \leq \frac{\gamma(\beta+1-\delta)}{2[\delta+\gamma(2\beta+1-\delta)](1+\lambda)} .$$

Thus

$$|f(z)| \leq |z|+\frac{\gamma(\beta+1-\delta)}{2[\delta+\gamma(2\beta+1-\delta)](1+\lambda)}|z|^2 .$$

$$|f(z)| \geq |z|-\sum_{k=2}^{\infty}a_k|z|^k$$

$$\geq |z|-|z|^2\sum_{k=2}^{\infty}a_k$$

$$|f(z)| \geq |z|-\frac{\gamma(\beta+1-\delta)}{2[\delta+\gamma(2\beta+1-\delta)](1+\lambda)}|z|^2 .$$

Theorem 4: If$f \in M(\delta,\beta,\gamma,\lambda)$, at that time

$$1-\frac{\gamma(\beta+1-\delta)}{[\delta+\gamma(2\beta+1-\delta)](1+\lambda)}|z| \leq |f'(z)| \leq 1+\frac{\gamma(\beta+1-\delta)}{[\delta+\gamma(2\beta+1-\delta)](1+\lambda)}|z| . \quad (14)$$

The finding is sharp for the $f(z)$ function offered through

$$f(z)=z+\frac{\gamma(\beta+1-\delta)}{2[\delta+\gamma(2\beta+1-\delta)](1+\lambda)}z^2 .$$

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Evidence:

$$|f'(z)| = 1 + \sum_{k=2}^{\infty} k a_k |z|^{k-1}$$

$$\leq 1 + 2|z| \sum_{k=2}^{\infty} a_k .$$

ThroughTheorem1

$$\sum_{k=2}^{\infty} a_k \leq \frac{\gamma(\beta + 1 - \delta)}{2[\delta + \gamma(2\beta + 1 - \delta)](1 + \lambda)} .$$

Thus

$$|f'(z)| \leq 1 + \frac{\gamma(\beta + 1 - \delta)}{[\delta + \gamma(2\beta + 1 - \delta)](1 + \lambda)} |z|$$

$$|f'(z)| \geq 1 - \sum_{k=2}^{\infty} k a_k |z|^{k-1}$$

$$\geq 1 - 2|z| \sum_{k=2}^{\infty} a_k$$

$$|f'(z)| \geq 1 - \frac{\gamma(\beta + 1 - \delta)}{[\delta + \gamma(2\beta + 1 - \delta)](1 + \lambda)} |z| \ ,$$

This completes the evidence.

5. Star likeness Radii, Convexity

In the theorems as follow, we have the star likeness radii, convexity and close- to-convexity for $M(\delta, \beta, \gamma, \lambda)$ class.

Theorem (5): If$f(z) \in M(\delta, \beta, \gamma, \lambda, r)$, at that time $f(z)$ is univalent star like in$|z| < r_1$, of ρ $(0 \leq \rho < 1)$order, in which

$$r_1 = \inf_k \left[\frac{k(1 - \rho)[\delta(k-1) + \gamma(\beta(k-1) + \beta + 1 - \delta)]B_k(\lambda)}{\gamma(k - \rho)(\beta + 1 - \delta)} \right]^{\frac{1}{k-1}} \qquad (15)$$

The finding is sharp for the $f(z)$ function offered through

$$f(z) = z + \frac{\gamma(\beta + 1 - \delta)}{k[\delta(k-1) + \gamma(\beta(k-1) + \beta + 1 - \delta)]B_k(\lambda)} z^k .$$

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Evidence: We should display that

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1-\rho \quad ,$$

Indeed, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}a_k|z|^{k-1}} \le 1-\rho$$

If

$$\frac{(k-\rho)|z|^{k-1}}{(1-\rho)} \le \frac{k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)}{\gamma(\beta+1-\delta)}$$

We get

$$|z|^{k-1} \le \frac{k(1-\rho)[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)}{\gamma(k-\rho)(\beta+1-\delta)}$$

Therefore

$$|z| \le \left[\frac{k(1-\rho)[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)}{\gamma(k-\rho)(\beta+1-\delta)}\right]^{\frac{1}{k-1}}.$$

Set $|z| = r_1$, we obtain the anticipated finding.

Theorem 6: If $f(z) \in M(\delta,\beta,\gamma,\lambda,r)$, At that time $f(z)$ is univalent convex of order $\rho(0 \le \rho < 1)$ in $|z| < r_2$, in which

$$r_2 = \inf_k \left[\frac{(1-\rho)[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)}{\gamma(k-\rho)(\beta+1-\delta)}\right]^{\frac{1}{k-1}} ,k \ge 2 \qquad (16)$$

The finding is sharp for the $f(z)$ function offered through

$$f(z) = z + \frac{\gamma(\beta+1-\delta)}{k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)} z^k .$$

Evidence: We should display that

$$\left|\frac{zf''(z)}{f'(z)}-1\right|<1-\rho \quad ,$$

Indeed, we have

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{\sum_{k=2}^{\infty} k(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty} ka_k|z|^{k-1}} \leq 1-\rho$$

If

$$\frac{k(k-\rho)|z|^{k-1}}{(1-\rho)} \leq \frac{k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)}{\gamma(\beta+1-\delta)}$$

$$|z|^{k-1} \leq \frac{(1-\rho)[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)}{\gamma(k-\rho)(\beta+1-\delta)}.$$

Therefore

$$|z| \leq \left[\frac{(1-\rho)[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)}{\gamma(k-\rho)(\beta+1-\delta)}\right]^{\frac{1}{k-1}}.$$

Setting $|z| = r_2$, we get the desired finding .

Theorem 7: Assume $f_1(z), f_2(z), \dots, f_l(z)$ specified through

$$f_s(z) = z + \sum_{k=2}^{\infty} a_{k,s} z^k \ ,\left(a_{k,s} \geq 0, \quad s = 1,2,\dots,l, \quad k \geq 2\right) \tag{17}$$

As it is in the $M(\delta,\beta,\gamma,\lambda)$ class. At that time the arithmetic mean of $f_s(z)$ $(i = 1,2,\dots,l)$ specified through

$$k(z) = \frac{1}{l}\sum_{s=1}^{l} f_s(z) \tag{18}$$

Be as well in the $M(\delta,\beta,\gamma,\lambda)$class.

Evidence: Through (17), (18), we able to state

$$k(z) = \frac{1}{l}\sum_{s=1}^{l}\left(z + \sum_{k=2}^{\infty} a_{k,s} z^k\right)$$
$$= z + \sum_{k=2}^{\infty}\left(\frac{1}{l}\sum_{s=1}^{l} a_{k,s}\right) z^k.$$

Since $f_s \in M(\delta,\beta,\gamma,\lambda)$ for each $(s = 1,2,\dots,l)$ so through utilizing Theorem1, we verify that

$$\sum_{k=2}^{\infty} k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)\left(\frac{1}{l}\sum_{s=1}^{l} a_{k,s}\right)$$

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$$= \frac{1}{l}\sum_{s=1}^{l}\left(\sum_{k=2}^{\infty} k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)\, a_{k,s}\right)$$

$$\leq \frac{1}{l}\sum_{s=1}^{l}\gamma(\beta+1-\delta)$$

$$= \gamma(\beta+1-\delta).$$

6. Linear Combination:

Theorem 8: Assume

$$f_i(z) = z + \sum_{k=2}^{\infty} a_{k,i}\, z^k \ ,\left(a_{k,i} \geq 0,\ \ i = 1,2,\dots,l,\quad k \geq 2\right) \tag{19}$$

Fit in to the $M(\delta,\beta,\gamma,\lambda)$class. At that time

$$F(z) = \sum_{i=1}^{l} G_i\, f_i(z)\ \in M(\delta,\beta,\gamma,\lambda),$$

Where

$$\sum_{i=1}^{l} G_i = 1. \tag{20}$$

Evidence: Through Theorem1, we able to state for each $i \in \{1,2,\dots l\}$

$$\sum_{k=2}^{\infty}\frac{k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)}{\gamma(\beta+1-\delta)}a_{k,i} \leq 1,$$

Therefore

$$F(z) = \sum_{i=1}^{l} G_i\left(z + \sum_{k=2}^{\infty} a_{k,i}\, z^k\right)$$
$$= z + \sum_{k=2}^{\infty}\left(\sum_{i=1}^{l} G_i a_{k,i}\right) z^k.$$

However

$$\sum_{k=2}^{\infty}\frac{k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)}{\gamma(\beta+1-\delta)}\left(\sum_{i=1}^{l} G_i a_{k,i}\right)$$
$$= \sum_{i=1}^{l} G_i\left[\sum_{k=2}^{\infty}\frac{k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)}{\gamma(\beta+1-\delta)}a_{k,i}\right] \leq 1.$$

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At that time

$F(z) \in M(\delta, \beta, \gamma, \lambda)$. So, the evidence is complete.

5. Hadamard product:

Theorem (9): Assume

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

Fit in the $(\delta, \beta, \gamma, \lambda)$class. . At that time the product of Hadamard of $f \; and \; g$ is offered through"

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$$

belongs to $M(\delta, \beta, \gamma, \lambda)$,

In which

$$\mu = \frac{\delta\gamma^2(\beta + 1 - \delta)(k-1)}{k[\delta(k-1) + \gamma(\beta(k-1) + \beta + 1 - \delta)]^2 B_k - \gamma^2(\beta + 1 - \delta)(\beta(k-1) + \beta + 1 - \delta)}.$$

Evidence: Since $f \; and \; g$ are in the $M(\delta, \beta, \gamma, \lambda)$class, at that time

$$\sum_{k=2}^{\infty} \frac{k[\delta(k-1) + \gamma(\beta(k-1) + \beta + 1 - \delta)]B_k(\lambda)}{\gamma(\beta + 1 - \delta)} a_n \leq 1, \tag{21}$$

and

$$\sum_{k=2}^{\infty} \frac{k[\delta(k-1) + \gamma(\beta(k-1) + \beta + 1 - \delta)]B_k(\lambda)}{\gamma(\beta + 1 - \delta)} b_k \leq 1. \tag{22}$$

We should detect the largest μ as

$$\sum_{k=2}^{\infty} \frac{k[\delta(k-1) + \mu(\beta(k-1) + \beta + 1 - \delta)]B_k(\lambda)}{\mu(\beta + 1 - \delta)} a_k \, b_k \leq 1. \tag{23}$$

Through Cauchy-Schwarz inequality, we get

$$\sum_{k=2}^{\infty} \frac{k[\delta(k-1) + \gamma(\beta(k-1) + \beta + 1 - \delta)]B_k(\lambda)}{\gamma(\beta + 1 - \delta)} \sqrt{a_k b_k} \leq 1. \tag{24}$$

Therefore, it is sufficient to display that

$$\frac{k[\delta(k-1) + \mu(\beta(k-1) + \beta + 1 - \delta)]B_k(\lambda)}{\mu(\beta + 1 - \delta)} a_k b_k \leq \frac{k[\delta(k-1) + \gamma(\beta(k-1) + \beta + 1 - \delta)]B_k(\lambda)}{\gamma(\beta + 1 - \delta)} \sqrt{a_k b_k}.$$

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That is

$$\sqrt{a_k b_k} \leq \frac{\mu[\delta(k-1) + \gamma(\beta(k-1) + \beta + 1 - \delta)]}{\gamma[\delta(k-1) + \mu(\beta(k-1) + \beta + 1 - \delta)]}. \tag{25}$$

From (24)

$$\sqrt{a_k b_k} \leq \frac{\gamma(\beta + 1 - \delta)}{k[\delta(k-1) + \gamma(\beta(k-1) + \beta + 1 - \delta)]B_k(\lambda)}. \tag{26}$$

Therefore, it is sufficient to display that

$$\frac{\gamma(\beta + 1 - \delta)}{k[\delta(k-1) + \gamma(\beta(k-1) + \beta + 1 - \delta)]B_k(\lambda)} \leq \frac{\mu[\delta(k-1) + \gamma(\beta(k-1) + \beta + 1 - \delta)]}{\gamma[\delta(k-1) + \mu(\beta(k-1) + \beta + 1 - \delta)]} \tag{27}$$

which simplifies to

$$\mu \leq \frac{\delta\gamma^2(\beta + 1 - \delta)(k-1)}{k[\delta(k-1) + \gamma(\beta(k-1) + \beta + 1 - \delta)]^2 B_k - \gamma^2(\beta + 1 - \delta)(\beta(k-1) + \beta + 1 - \delta)}.$$

Theorem10: Assume the tasks f_j $(j = 1,2)$ specified through

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j}\, z^k, \ \ (a_{k,j} \geq 0\,, j = 1,2) \tag{28}$$

As it is in the$M(\delta, \beta, \gamma, \lambda)$ class . At that time the function h specified through

$$h(z) = z - \sum_{k=2}^{\infty} (a_{k,1}{}^2 + a_{k,2}{}^2)\, z^k, \tag{29}$$

belongs to the $M(\delta, \beta, \gamma, \lambda)$class." In which

$$\varphi = \frac{2\delta\gamma^2(k-1)(\beta + 1 - \delta)}{\gamma^2(\beta(k-1) + \beta + 1 - \delta)[k(\beta(k-1) + \beta + 1 - \delta) - 2(\beta + 1 - \delta)] + k\delta^2(k-1)^2}.$$

Evidence: We should detect the largest φ as

$$\sum_{k=2}^{\infty} \frac{k[\delta(k-1) + \varphi(\beta(k-1) + \beta + 1 - \delta)]B_k(\lambda)}{\varphi(\beta + 1 - \delta)} \left(a_{k,1}{}^2 + a_{k,2}{}^2\right) \leq 1. \tag{30}$$

Since $f_j(z) \in M(\delta, \beta, \gamma, \lambda)(j = 1,2)$, we get

$$\sum_{k=1}^{\infty}\left(\frac{k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)}{\gamma(\beta+1-\delta)}\right)^2 {a_{k,1}}^2 \leq \left(\sum_{k=1}^{\infty}\frac{k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)}{\gamma(\beta+1-\delta)}a_{k,1}\right)^2 \leq 1 \quad (31)$$

and

$$\sum_{k=1}^{\infty}\left(\frac{k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)}{\gamma(\beta+1-\delta)}\right)^2 {a_{k,2}}^2 \leq \left(\sum_{k=1}^{\infty}\frac{k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)}{\gamma(\beta+1-\delta)}a_{k,2}\right)^2 \leq 1 \quad (32)$$

Combining the inequalities (31) and (32), gives

$$\sum_{k=1}^{\infty}\frac{1}{2}\left(\frac{k[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)}{\gamma(\beta+1-\delta)}\right)^2 \left({a_{k,1}}^2+{a_{k,2}}^2\right) \leq 1. \qquad (33)$$

But, $h \in M(\delta,\beta,\gamma,\lambda)$ if and single if

$$\sum_{k=2}^{\infty}\frac{k[\delta(k-1)+\varphi(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)}{\varphi(\beta+1-\delta)}\left({a_{k,1}}^2+{a_{k,2}}^2\right) \leq 1 \qquad (34)$$

The inequality (34) would be satisfied if

$$\frac{k[\delta(k-1)+\varphi(\beta(k-1)+\beta+1-\delta)]B_k(\lambda)}{\varphi(\beta+1-\delta)} \leq \frac{k^2[\delta(k-1)+\gamma(\beta(k-1)+\beta+1-\delta)]^2B_k(\lambda)}{2\gamma^2(\beta+1-\delta)^2} \qquad (35)$$

Hence

$$\varphi = \frac{2\delta\gamma^2(k-1)(\beta+1-\delta)}{\gamma^2(\beta(k-1)+\beta+1-\delta)[k(\beta(k-1)+\beta+1-\delta)-2(\beta+1-\delta)]+k\delta^2(k-1)^2}.$$

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