JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 5462-5469 https://publishoa.com
ISSN: 1309-3452

Approximate analytical solution of one dimensional nonlinear Burger's equation using Homotopy Perturbation method

Diwari Lal

Associate Professor, Department of Mathematics, Narain (PG) College, Shikohabad
(affiliated to Dr. B.R.A. University), Agra (U.P.), India

Manoj Yadav

Research Scholar, Department of Mathematics, Narain (PG) College, Shikohabad
(affiliated to Dr. B.R.A. University), Agra (U.P.), India
Email id of corresponding Author: yadavmanoj265@gmail.com

Received 2022 March 15; **Revised** 2022 April 20; **Accepted** 2022 May 10.

Abstract- *As a solution to the nonlinear Burger's issue in a single dimension, the homotopy perturbation method (HPM) is what we recommend utilizing according to the findings of this research. We have reached a series solution of the equations in terms of convergent series with easily computable components thanks to the fact that the nonlinear elements of Burger's equations may be addressed by using the homotopy perturbation approach (HPM). HPM is closely associated with the concept of the sum of an infinite series term, which frequently and rapidly converges to the correct response. The intricate equation is simplified into a more understandable form by using the HPM. According to the data that was gathered, the method that was recommended performs better than the state-of-the-art solutions for similar PDEs, and it is also much easier to put into practice.*
Keywords-Homotopy Perturbation Method, Burgers' Equation, Convergent Series

1. Introduction-

According to studies, nonlinear equations are among the most significant occurrences in the world. The fields of applied mathematics, physics, and engineering all feel the influence of non linear processes. This means that various causes contribute to the change of each parameter. There is still a significant demand for novel approaches to finding exact or approximate solutions to nonlinear partial differential equations in physics and mathematics. There is typically no exact analytic solution to a nonlinear problem. Additionally, nonlinear equations can be addressed using some analytic methods. Perturbation techniques and the d-expansion method are two examples of time-tested, conventional analytic approaches. Solutions to nonlinear partial differential equations have been the focus of research by a number of writers in recent years.

To simulate several fascinating situations in practical mathematics, Burger's equation is a useful tool. Some fluid flow problems, such as those involving shocks or viscous dissipation, are well-represented by this model. Burger's equation was named after his comprehensive studies, and the first steady-state solutions were provided by Bateman (1915). Even though Burger (1939) developed the equation to simulate turbulence, it may be used to a wide variety of other physical phenomena, including shock flows, traffic flow, acoustic transmission in fog, etc. In 2003, HE provided an in-depth analysis of the history of the fourth-order fractal derivative non-linear integral boundary value problem.

Momani (2006) modified the 1-D Burgers equation by exchanging the time and space derivatives for the Caputo fractional derivative. This was done in order to make the equation more computationally efficient. Analytical solutions to the generalized Burgers problem were established by him through the use of the ADM. Yildirim (2009) presented the concept of HPM with the intention of mathematically resolving the fractional PDE. Biazar and Eslami (2011) are the ones responsible for the introduction of the new Homotopy perturbation approach for solving partial differential equations. Mittal and Jiwari (2012) were successful in their attempt to solve the coupled viscous Burgers' equations by employing the DQM. Gomez (2014) examined the Jumeaux-refined Riemann-Liouville fractional derivative Burgers equation for the purpose of his article. He was successful in solving the fractional Burgers equation by employing a technique known as the fractional complex transform. Yang and Wang (2019) implemented the concept of a local fractional HPM for the purpose of finding a non-homogeneous solution to the local fractional Korteweg-De Vries problem. Analytical approximation progressive wave solutions are presented in Demiray and Zahar's (2020) research. Cheng and Bang (2021) came up with a method that uses fourth-order finite differences and keeps the energy dissipation intact.

2. Homotopy Perturbation Theory: The Core Concept-

A semi-analytical method known as the HPM can be used to solve linear and nonlinear ODE and PDE. He (1999) came up with the idea for the HPM.

Here is a differential equation to think about:

$$B(\theta) - g(t) = 0, t \epsilon t_1 \quad (1)$$

according to the boundary conditions

$$C\left(\theta, \frac{\partial \theta}{\partial t}\right) = 0, t \epsilon t_2 \quad (2)$$

where B stands for the general differential operator, C stands for the boundary operator, $t_2$stands for the endpoint of the domain t_1 and $g(t)$ is a well-known analytic function.

The operator B can be broken down into its linear (P) and nonlinear components (Q). Therefore, the following form of Eq. (1) is possible:

$$P(\theta) + Q(\theta) - g(t) = 0 \quad (3)$$

Inserting an arbitrary parameter r into Eq. (3) yields the following:

$$P(\theta) + r[Q(\theta) - g(t)] = 0 \quad (4)$$

where the embedding parameter r is an integer between zero and one.

By doing this, we are able to create the homotopy, Applying the condition $w(t,r): t_1 \times [0,1] \rightarrow R$ in Eq. (3), we have

$$H(w,r) = (1-r)[P(w) - P(\theta_0)] + r[P(w) + Q(w) - g(t)]$$
$$\Rightarrow H(w,r) = P(w) - P(\theta_0) + rP(\theta_0) + r[Q(w) - g(t)] \quad (5)$$

Specificallyθ_0 is a first approximation of Eq. (5) that meets the requirements stated above.

The following equations may be obtained by replacing $r = 0$ and $r = 1$ into Eq. (6).

$$H(w,0) = P(w) - P(\theta_0)$$
$$H(w,1) = P(w) - P(\theta_0) + P(\theta_0) + [Q(w) - g(t)] = P(w) + Q(w) - g(t)$$
$$= B(w) - g(t) = 0$$

How about we look at the answer to Eq. (5) as a power series in r as shown below?

$$w = w_0 + rw_1 + r^2 w_2 + r^3 w_3 + \cdots \quad (6)$$

Afterward, we can get a close approximation to the answer to Eq. (1) by solving for

$$\theta(x,t) = \lim_{r \rightarrow 1} w = w_0 + w_1 + w_2 + w_3 + \cdots \quad (7)$$

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 5462-5469 https://publishoa.com
ISSN: 1309-3452

3. The solution of the one-dimensional Burger equation via the application of the Homotopy Perturbation technique (HPM)-

Problem-3.1: Let us think about the Burgers equation in one dimension
$\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{dx} = \frac{\partial^2\theta}{\partial x^2}$, $0 < x < 1, t > 0$ (8)

With initial condition
$\theta(x,0) = 4x(1-x)$ (9)

With the use of HPM, we develop the following homotopy for solving the 1D convection diffusion
$\frac{\partial\theta}{\partial t} - \frac{\partial\theta_0}{\partial t} = r\left(\frac{\partial^2\theta}{\partial x^2} - \frac{\partial\theta}{\partial t} - u\frac{\partial\theta}{dx}\right)$ (10)

and the initial approximations are as follow
$\theta_0(x,t) = \theta(x,0) = 4x(1-x)$ (11)
Coefficient of r^0: $\frac{\partial\theta_0}{\partial t} - \frac{\partial\theta_0}{\partial t} = 0$ (12)
Coefficient of r^1: $\frac{\partial\theta_1}{\partial t} = \frac{\partial^2\theta_0}{\partial x^2} - \frac{\partial\theta_0}{\partial t} - \theta_0\frac{\partial\theta_0}{dx}$ (13)
Coefficient of r^2: $\frac{\partial\theta_2}{\partial t} = \frac{\partial^2\theta_1}{\partial x^2} - \frac{\partial\theta_1}{\partial t} - \theta_0\frac{\partial\theta_1}{dx} - \theta_1\frac{\partial\theta_0}{dx}$ (14)

By choosing $u_0(x,t) = u(x,0) = 4x(1-x)$, and solving the above equations, we obtain the following approximation-
$\theta_0(x,t) = \theta(x,0) = 4x - 4x^2$ (15)
$\theta_1 = (-32x^3 + 48x^2 - 16x - 8\,)t$ (16)
$\theta_2 = (32x^3 - 112x^2 + 80x + 8)t + (128x^4 - 256x^3 + 160x^2 - 96x + 32)t^2$ (17)

The solution of equation (1) is given as
$\theta(x,t) = \theta_0 + \theta_1 + \theta_2 + \theta_3 + \cdots$
$\theta(x,t) = 4x - 4x^2 + (-32x^3 + 48x^2 - 16x - 8\,)t\ + (32x^3 - 112x^2 + 80x + 8)t + (128x^4 - 256x^3 + 160x^2 - 96x + 32)t^2 + \cdots$
$\theta(x,t) = 4x - 4x^2 + (-32x^3 + 48x^2 - 16x - 8 + 32x^3 - 112x^2 + 80x + 8)t\ + (128x^4 - 256x^3 + 160x^2 - 96x + 32)t^2 + \cdots$
$\theta(x,t) = 4x - 4x^2 - (64x^2 - 64x)t\ + (128x^4 - 256x^3 + 160x^2 - 96x + 32)t^2$
$\theta(x,t) = 4x(1-x) - 64x(x-1)t\ + 32(4x^4 - 8x^3 + 5x^2 - 3x + 1)t^2$ (18)

Problem-3.2: Let us think about the Burgers equation in one dimension
$\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{dx} = \frac{\partial^2\theta}{\partial x^2}$, $0 < x < 1, t > 0$ (19)
With initial condition
$\theta(x,0) = sin\pi x$ (20)

With the use of HPM, we develop the following homotopy for solving the 1D convection diffusion
$\frac{\partial\theta}{\partial t} - \frac{\partial\theta_0}{\partial t} = r\left(\frac{\partial^2\theta}{\partial x^2} - \frac{\partial\theta}{\partial t} - u\frac{\partial\theta}{dx}\right)$ (21)
and the initial approximations are as follow
$\theta_0(x,t) = \theta(x,0) = sin\pi x$ (22)

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 5462-5469 https://publishoa.com
ISSN: 1309-3452

Coefficient of r^0: $\frac{\partial\theta_0}{\partial t} - \frac{\partial\theta_0}{\partial t} = 0$ (23)

Coefficient of r^1: $\frac{\partial\theta_1}{\partial t} = \frac{\partial^2\theta_0}{\partial x^2} - \frac{\partial\theta_0}{\partial t} - \theta_0 \frac{\partial\theta_0}{dx}$ (24)

Coefficient of r^2: $\frac{\partial\theta_2}{\partial t} = \frac{\partial^2\theta_1}{\partial x^2} - \frac{\partial\theta_1}{\partial t} - \theta_0 \frac{\partial\theta_1}{dx} - \theta_1 \frac{\partial\theta_0}{dx}$ (25)

By choosing $u_0(x,t) = u(x,0) = 4x(1-x)$ (26)

$\theta_1 = -(\pi^2 sin\pi x + \pi sin\pi x cos\pi x)t$ (27)

$\theta_2 = (\pi^2 sin\pi x + \pi sin\pi x cos\pi x)t + (\pi^4 sin\pi x + 6\pi^3 sin\pi x cos\pi x + \pi^2 sin^3 \pi x)\frac{t^2}{2}$ (28)

$\theta(x,t) = \theta_0 + \theta_1 + \theta_2 + \theta_3 + \cdots$

$\theta(x,t) = sin\pi x + (\pi^4 sin\pi x + 6\pi^3 sin\pi x cos\pi x + \pi^2 sin^3 \pi x)\frac{t^2}{2} + \cdots$ (29)

Problem 3.3- Let us think about the Burgers equation in one dimension

$\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{dx} = \frac{\partial^2\theta}{\partial x^2}, 0 < x < 1, t > 0$ (30)

With initial condition

$\theta(x,0) = e^x$ (31)

With the use of HPM, we develop the following homotopy for solving the 1D convection diffusion

$\frac{\partial\theta}{\partial t} - \frac{\partial\theta_0}{\partial t} = r\left(\frac{\partial^2\theta}{\partial x^2} - \frac{\partial\theta}{\partial t} - u\frac{\partial\theta}{dx}\right)$ (32)

and the initial approximations are as follow

$\theta_0(x,t) = \theta(x,0) = e^x$ (33)

Coefficient of r^0: $\frac{\partial\theta_0}{\partial t} - \frac{\partial\theta_0}{\partial t} = 0$ (34)

Coefficient of r^1: $\frac{\partial\theta_1}{\partial t} = \frac{\partial^2\theta_0}{\partial x^2} - \frac{\partial\theta_0}{\partial t} - \theta_0 \frac{\partial\theta_0}{dx}$ (35)

Coefficient of r^2: $\frac{\partial\theta_2}{\partial t} = \frac{\partial^2\theta_1}{\partial x^2} - \frac{\partial\theta_1}{\partial t} - \theta_0 \frac{\partial\theta_1}{dx} - \theta_1 \frac{\partial\theta_0}{dx}$ (36)

$\theta_0(x,t) = \theta(x,0) = e^x$ (37)

$\theta_1 = (e^x - e^{2x})t$ (38)

$\theta_2 = (e^{2x} - e^x)t + (e^x - 6e^{2x} + 2e^{3x} + 3e^{3x})\frac{t^2}{2}$ (39)

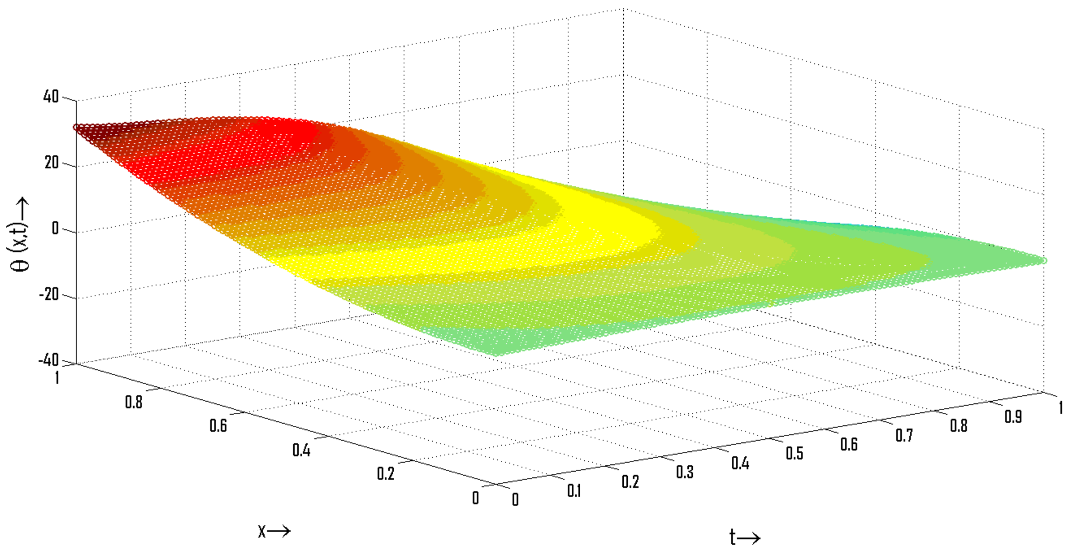
$\theta(x,t) = \theta_0 + \theta_1 + \theta_2 + \theta_3 + \cdots$

$\theta(x,t) = e^x + (e^x - 6e^{2x} + 2e^{3x} + 3e^{3x})\frac{t^2}{2} + \cdots$ (40)

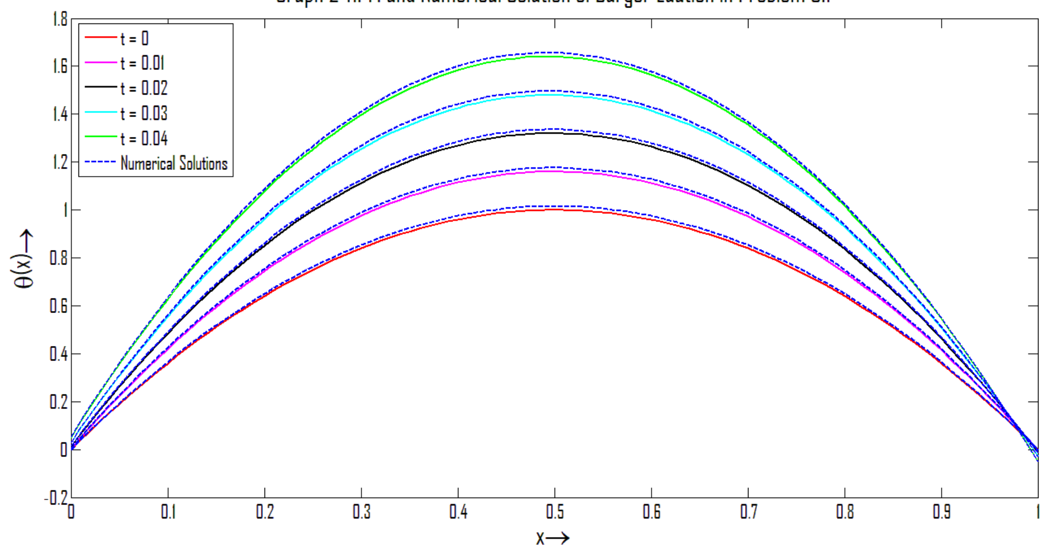
JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 5462-5469 https://publishoa.com
ISSN: 1309-3452

4. Mathematical Outcomes and Discussion-

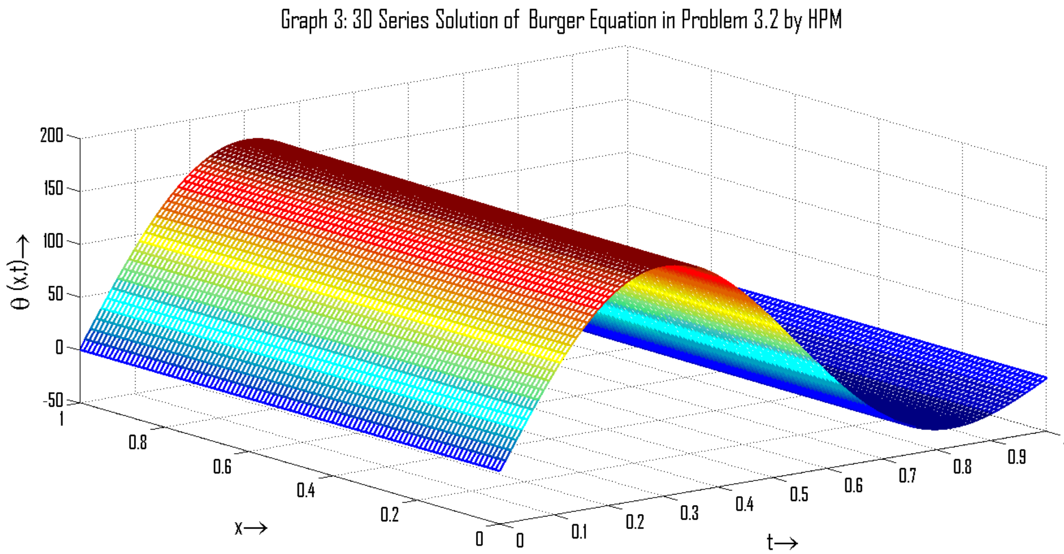
Graph 1: 3D Series Solution of Burger Equation in Problem 3.1 by HPM

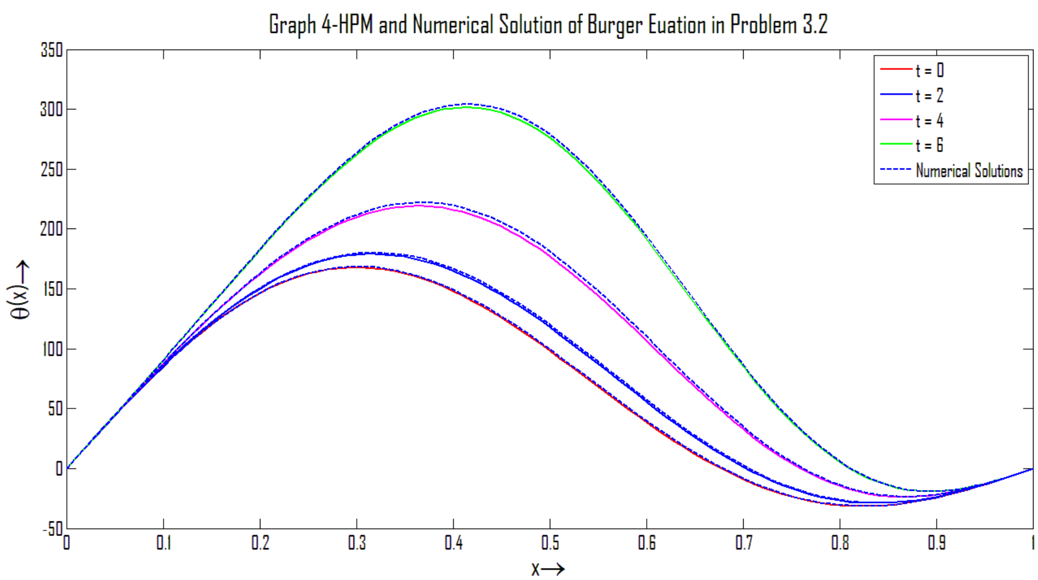


Graph 2-HPM and Numerical Solution of Burger Euation in Problem 3.1



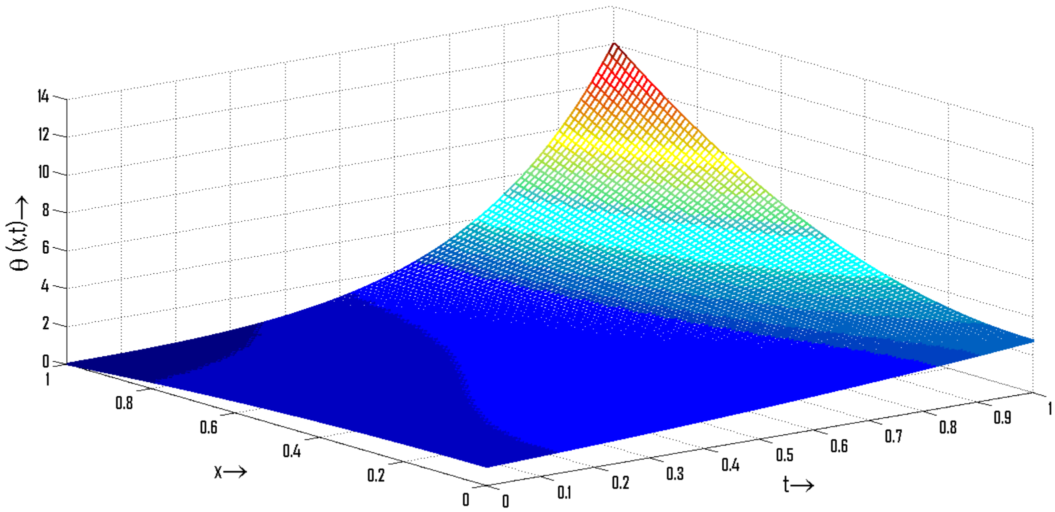
JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 5462-5469 https://publishoa.com
ISSN: 1309-3452



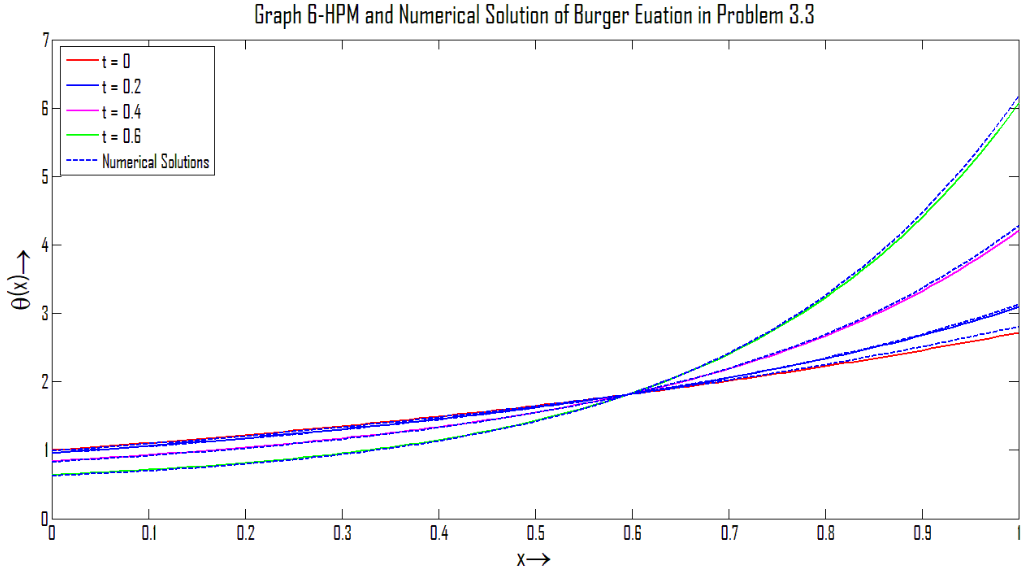


JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 5462-5469 https://publishoa.com
ISSN: 1309-3452

Graph 5: 3D Series Solution of Burger Equation in Problem 3.3 by HPM



Graph 6-HPM and Numerical Solution of Burger Euation in Problem 3.3



Using HPM, the 3D approximate solution to problems (3.1) to (3.3) has been displayed in graphs (1), (3), and (5) respectively. It is clear from looking at graphs (2), (4), and (6) that the solution to these problems that is produced by using the numerical solution is extremely similar to the approximate solution that is obtained using the HPM. The homotopy perturbation method allows us to acquire an approximation solution, and if we compare this to the numerical solution, we can see that they are consistent with one another.

5. Concluding Remarks-

For the purpose of this research, we pondered applying HPM to the problem of resolving Burgers' equation. He's HPM is a powerful basic method. Our new, effective recurrent relation for resolving the nonlinear Burgers' problem was derived using this strategy. He's HPM is apt to be utilized as an alternative strategy to existing techniques being utilized to a wide variety of physical difficulties. A comparison is made between the approximate analytical solutions obtained and the numerical solution. The examples (3.1)-(3.3) were explored to show that the outcomes of this method have a high degree of congruence with the outcomes produced via numerical solution. A significant number of mathematicians are actively conducting research in the field of convergence analysis of the technique.

REFERENCES-

1. Bateman (1915): “Some recent researches on the motion of fluids”, Monthly weather Review, 43:163-170.
2. Biazar J. and Eslami M. (2011): “A new homotopy perturbation method for solving systems of partial differential equations”, Computers and Mathematics with Application, 62:225–234
3. Burger J. M. (1939): “Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion”, Transaction Royal Netherland Academy Science, 17:1-53.
4. Cheng H., Wang X. (2021): “A high-order linearized difference scheme preserving dissipation property for the 2D Benjamin-Bona-Mahony-Burgers equation”, Journal of Mathematical Analysis and Applications, 500(2): 125182
5. Demiray H.,EI Zahar E.R. (2020): “Analytical approximate solutions for nonplanar Burgers equations by weighted residual method”, Results in Physics, 14:103293
6. Gomez S. C.A. (2014): “A note on the exact solution for the fractional Burgers equation”, International Journal of pure and applied mathematics, 93(2):229–232
7. He J.H. (1999): “Homotopy perturbation technique. Computer Methods in Applied Mechanics and Engineering”, 178 (3): 257–262
8. He J.H. (2003): “Homotopy perturbation method: a new nonlinear analytical technique” Applied Mathematics and Computations, 135:73–79
9. Mittal R.C., Jiwari R. (2012): “Differential quadrature method for numerical solution of coupled viscous burgers’ equations”, International Journal of Computational Methods in Engineering Science and Mechanics,13:88–92
10. Momani S. (2006): “Non-perturbative analytical solutions of the space- and time-fractional Burgers equations”, Chaos Solitons Fractals, 28(4):930–937
11. Yang Y.J.,Wang S.Q. (2019): “A local fractional homotopy perturbation method for solving the local fractional korteweg-de vries equations with non-homogeneous term”; Thermal Science, 23:1495–1501
12. Yildirim A. (2009): “On the solution of the nonlinear korteweg-de vries equation by the homotopy perturbation method”, Communications in Numerical Methods in Engineering , 25:1127–1136