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Analytical Solution of the Chaotic Piecewise Linear Planar Map

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Abstract

Generally, there is no analytical method to calculate the chaotic solution of a dynamical system. In this paper, a simple proof of the existence of analytical solution in a discontinuous 2D discrete time piecewise chaotic map is reported, i.e., we show that a map that has a simple analytical solution can also have a chaotic attractor by the use of the classical definition of the Jordan canonical form defined for matrices available in most kinds of literature on linear algebra.

Keywords Piecewise maps, fixed points, chaotic attractors, analytical solution, linear algebra.

1. Introduction

Many physical and engineering systems have been found to be best represented by piecewise-linear or nonlinear chaotic maps. See for example [1, 2, 3, 4, 5] where the discrete-time state space is divided into two or more linear or nonlinear regions with different functional forms separated by borderlines. On the other hand the chaotic map cannot be studied with the classical methods of continuous system with analytic solutions, since such chaotic map must exhibit sensitive dependence to initial conditions. This idea of sensitivity dependence is otherwise called butterfly effect [6, 7, 8, 9, 10, et al.,]. On the other hand, the study of chaotic behaviors of a discrete mapping is a very interesting field in dynamical systems theory. For example [11, 12, 13 et al.,]. Thus, the purpose of this work is to show that a discontinuous 2D discrete time piecewise chaotic map that has a chaotic attractor can be also have a simple analytical solution. The research result is obtained by using the Jordan canonical form defined for matrices [14, 15]. The evolution of the attractors of the 2D discrete time piecewise chaotic map are analyzed by using the bifurcation diagram, the Lyapunov exponent diagram and phase portraits when the bifurcation parameters are varied.

2. The piecewise chaotic map

The 2-D non-invertible discrete piecewise map is given as follows

$$f(x_n, y_n) = \begin{pmatrix} -a|x_n| + y_n \\ bx_n + y_n \end{pmatrix} \tag{1}$$

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Where $(x,y) \in \mathbb{R}^2$ is the state space, n is the discrete time and $(a,b) \in \mathbb{R}^2$ are bifurcations parameters (are the interactions coefficients of the map (1)). This equation is a piecewise linear version of the homogenous linear map.

Map (1) can be rewritten as follows

$$f(x_n, y_n) = \begin{cases} \begin{pmatrix} -ax_n + y_n \\ bx_n + y_n \end{pmatrix} & \text{if } x_n \geq 0 \\ \begin{pmatrix} ax_n + y_n \\ bx_n + y_n \end{pmatrix} & \text{if } x_n < 0 \end{cases}$$

The only trivial fixed point of the system (1) is $A(0,0)$. The Jacobi matrix of the map (1) evaluated at a point (x,y) is given as

$$\begin{pmatrix} \mp a & 1 \\ b & 1 \end{pmatrix}$$

The characteristic polynomial of the Jacobi matrix of the map (1) calculated at the fixed point which takes the form

$\lambda^2 - (1-a)\lambda - (a+b) = 0$ if $x \geq 0$
$\lambda^2 - (1+a)\lambda + (a-b) = 0$ if $x < 0$

If $x \geq 0$, according to the Schur-Cohn-Jury criterion available in [16] the fixed point $A(0,0)$ of the map (1) is asymptotic stable if the following conditions are satisfied

$$b < 0, 2 - 2a - b > 0, 1 + a + b > 0$$

This leads to

$$b < 0, -1 - b < a < \frac{2-b}{2}$$

For $a = -0.4$ and $b = -0.5$ the eigenvalues are $\lambda_1 = 0.7 - 0.64031i$ and $\lambda_2 = 0.7 + 0.64031i$ therefore, the fixed point is asymptotically stable, thus $|\lambda i(1 \leq i \leq 2)| < 1$.

If $x < 0$, according to the Schur-Cohn-Jury criterion available in [16] the fixed point $A(0,0)$ of the map (1) is asymptotic stable if the following conditions are satisfied

$$b < 0, 2 + 2a - b > 0, 1 - a + b > 0$$

This leads to

$$b < 0, \frac{b-2}{2} < a < 1 + b$$

For $a = 0.4$ and $b = -0.5$ the eigenvalues are $\lambda_1 = 0.7 - 0.64031i$ and $\lambda_2 = 0.7 + 0.64031i$ therefore, the fixed point is asymptotically stable, thus $|\lambda i(1 \leq i \leq 2)| < 1$.

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3. Numerical results

In this section, we will illustrate some observed chaotic attractors, the dynamical behaviors of the map (1) are investigated numerically. Fig. 1 and Fig. 2 shows respectively the bifurcation diagram and the diagram of the variation of Lyapunov exponent of the map (1) by varying the parameter a. For the range -2 ≤ a ≤ -1. It can be observed from Fig. 1 that the map (1) undergoes the following dynamical behaviors as a increases:

For -2 < a ≤ -1.1, map (1) is chaotic. If we fix the parameter a to the value a = -1.80 and b = -0.003, the dynamical behavior of the map (1) is chaotic, which is verified by the corresponding largest Lyapunov exponent is positive, as shown in Fig. 2. The corresponding chaotic attractor is shown in Fig. 3. Also, Fig. 4 shows chaotic attractor of the map (1) (a = -2 and b = -0.003), Fig. 5 shows Chaotic attractor of the map (1) (a = -1.3 and b = -0.004), and Fig. 6 shows Chaotic attractor of the map (1) (a = -1.12 and b = -0.004).

For -1.1 < a ≤ -1, map (1) is periodic and there are several chaotic windows.

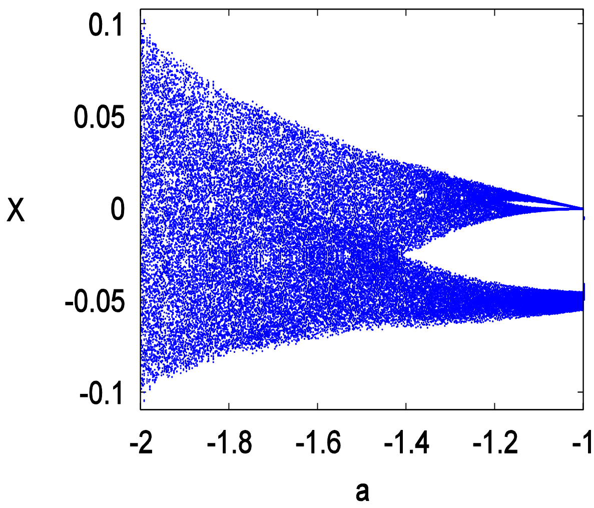

Fig. 1: Bifurcation diagram from map (1) for $-2 \le a \le -1$.

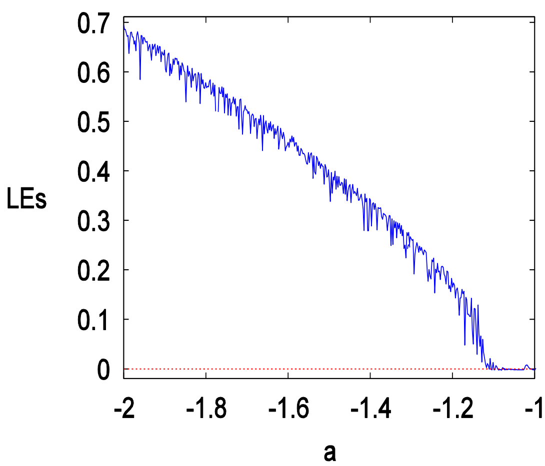

Fig. 2: Lyapunov exponents from map (1) for $-2 \le a \le -1$.

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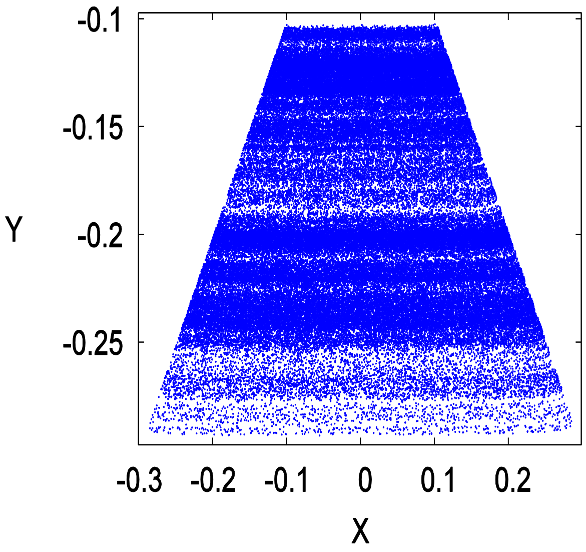

Fig. 3: Chaotic attractor from map (1) for $a = -2, b = -0.003$.

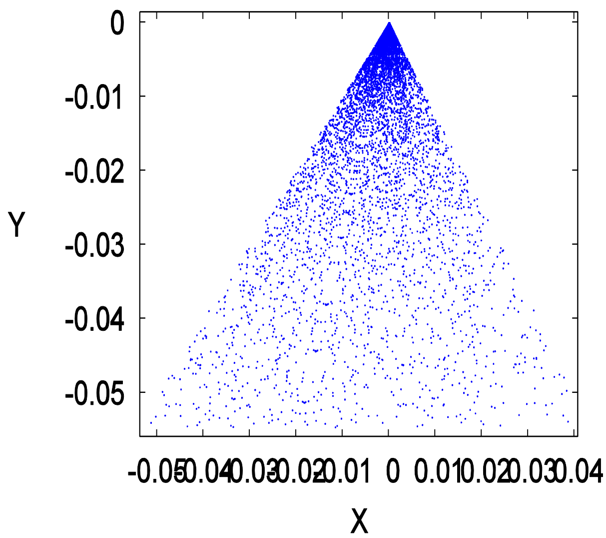

Fig. 4: Chaotic attractor from map (1) for $a = -1.8, b = -0.003$.

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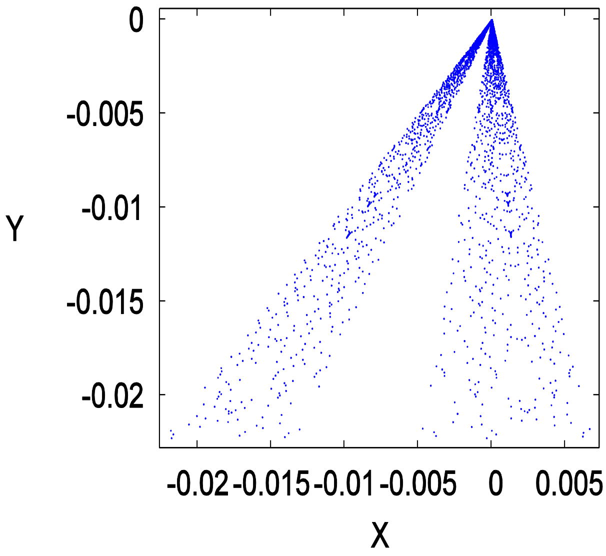

Fig. 5: Chaotic attractor from map (1) for $a = -1.3, b = -0.004$.

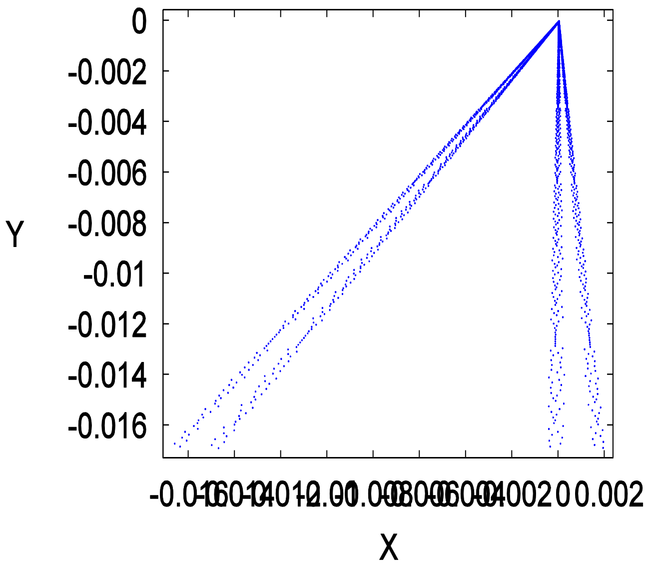

Fig. 6: Chaotic attractor from map (1) for $a = -1.12, b = -0.004$.

4. Analytical solution of the piecewise chaotic map

The map (1) has different periodic and chaotic attractors when changing the parameter a. The positive largest Lyapunov exponent indicates that the solution is chaotic see Fig. 2. Now we show that a discrete-time chaotic piecewise linear map (1) that has an analytical solution by using the Jordan canonical form defined for matrices exist in toolbox on linear algebra.

Map (1) can be written as

$$z_{n+1} = \begin{cases} Az_n & \text{if } x_n \geq 0 \\ Bz_n & \text{if } x_n < 0 \end{cases}$$

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Where

$$A = \begin{pmatrix} -a & 1 \\ b & 1 \end{pmatrix}, B = \begin{pmatrix} a & 1 \\ b & 1 \end{pmatrix} \text{ and } z_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Such analytical unique solution of map (1) has the form

$$z_{n+1} = \begin{cases} A^n z_n & \text{if } x_n \geq 0 \\ B^n z_n & \text{if } x_n < 0 \end{cases}, n = 0,1, \dots$$

Where

$$A^n = (PJP^{-1})^n = PJ^nP^{-1}$$

$$B^n = (\acute{P}\acute{J}\acute{P}^{-1})^n = \acute{P}J'^n\acute{P}^{-1}$$

And where J (resp $\acute{J}$) is the matrix whose columns consist of the two eigenvalues of the matrix A (resp B) and P (resp $\acute{P}$) is the matrix whose columns consist of the two eigenvectors of the matrix A (resp B).

Now, some algebra leads to the following from and we have

$$A^n = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$$

$$\begin{cases} \alpha_{11} = \dfrac{\frac{1}{4}\xi_1{}^{n-1}\xi_2\xi_3}{\xi_4} - \dfrac{\frac{1}{4}\xi_5{}^{n-1}\xi_2\xi_3}{\xi_4} \\ \alpha_{21} = \dfrac{\frac{1}{2}\xi_1{}^{n-1}\xi_3}{\xi_4} - \dfrac{\frac{1}{2}\xi_5{}^{n-1}\xi_2}{\xi_4} \\ \alpha_{12} = \dfrac{\frac{1}{4}\xi_1{}^{n}\xi_2\xi_3}{\xi_4} - \dfrac{\frac{1}{4}\xi_5{}^{n}\xi_2\xi_3}{\xi_4} \\ \alpha_{22} = \dfrac{\frac{1}{2}\xi_1{}^{n}\xi_3}{\xi_4} - \dfrac{\frac{1}{2}\xi_5{}^{n}\xi_2}{\xi_4} \end{cases}$$

and

$$\begin{cases} \xi_1 = \dfrac{1}{2} - \dfrac{1}{2}a + \dfrac{1}{2}\sqrt{(1+a)^2 + 4b} \\ \xi^2 = -1 + a + \dfrac{1}{2}\sqrt{(1+a)^2 + 4b} \\ \xi_3 = 1 - a + \dfrac{1}{2}\sqrt{(1+a)^2 + 4b} \\ \xi_4 = \sqrt{(1+a)^2 + 4b} \\ \xi_5 = \dfrac{1}{2} - \dfrac{1}{2}a - \dfrac{1}{2}\sqrt{(1+a)^2 + 4b} \end{cases}$$

And where

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$$B^n = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$$

where

$$\begin{cases} \beta_{11} = \dfrac{\frac{1}{4}\xi_6{}^n\xi_7\xi_8}{\xi_9\xi_{10}} - \dfrac{\frac{1}{4}\xi_{11}{}^n\xi_7\xi_8}{\xi_{10}\xi_{12}} \\ \beta_{21} = \dfrac{\frac{1}{2}\xi_6{}^n\xi_7}{\xi_9\xi_{10}} - \dfrac{\frac{1}{2}\xi_5{}^n\xi_7}{\xi_{10}\xi_{12}} \\ \beta_{12} = \dfrac{\frac{1}{4}\xi_6{}^n\xi_7\xi_8}{\xi_{10}} - \dfrac{\frac{1}{4}\xi_{11}{}^n\xi_7\xi_8}{\xi_{10}} \\ \beta_{22} = \dfrac{\frac{1}{2}\xi_6{}^n\xi_7}{\xi_{10}} - \dfrac{\frac{1}{2}\xi_{11}{}^n\xi_8}{\xi_{10}} \end{cases}$$

and

$$\begin{cases} \xi_6 = \dfrac{1}{2} + \dfrac{1}{2}a + \dfrac{1}{2}\sqrt{(1-a)^2 + 4b} \\ \xi_7 = 1 + a + \sqrt{(1-a)^2 + 4b} \\ \xi_8 = -1 + a + \sqrt{(1-a)^2 + 4b} \\ \xi_9 = \dfrac{1}{2} - \dfrac{1}{2}a + \dfrac{1}{2}\sqrt{(1-a)^2 + 4b} \\ \xi_{10} = \sqrt{(1-a)^2 + 4b} \\ \xi_{11} = \dfrac{1}{2} + \dfrac{1}{2}a - \dfrac{1}{2}\sqrt{(1-a)^2 + 4b} \\ \xi_{12} = \dfrac{1}{2} - \dfrac{1}{2}a - \dfrac{1}{2}\sqrt{(1-a)^2 + 4b} \end{cases}$$

5. Conclusion

This paper is devoted to the rigorous proof of the existence of analytical solution in a discontinuous 2D discrete time piecewise chaotic map. This interesting property is determined by using the classical methods of the Jordan canonical form available in tool books on linear algebra.

References

1. S. Banerjee and G. C. Verghese (ed), Nonlinear phenomena in power electronics: Attractors, bifurcations, chaos, and nonlinear control (IEEE Press, New York, USA) (2001).
2. R. Rajaraman, I. DobsonI and S. Jalali, Nonlinear dynamics and switching time bifurcations of a thyristor controlled reactor circuit, IEEE Trans. Circuits & Systems-I 43, 1001-1006 (1996).
3. T. K. Tse, Complex Behavior of Switching Power Converters (CRC Press, Boca Raton, USA) (2003).

4. Johansson, M. K. J, Piecewise Linear Control Systems: A Computational Approach (Springer, Berlin, Heidelberg) (2003).
5. S. Banerjee, S. Parui, and A. Gupta, Dynamical effects of missed switching in current-mode controlled dc-dc converters, IEEE Trans. Circuits & Systems-II 51, 649-54 (2004).
6. J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey, On Devaney's definition of chaos, The American Mathematical Monthly, v. 99, no. 4, 332–334 (1992).
7. T. Y. Li and J. A. Yorke, Period three implies chaos, The American Mathematical Monthly, l. 82, no. 10, 985–992 (1975).
8. Rössler Otto E, A equation for continuous chaos, Physics Letters A, n0. 57, 397-398 (1976).
9. P. R. Kulkarni and V. C. Borkar, Period three cycle and chaos in a dynamical system, International Journal of Multidisciplinary Research and Development, v. 2, no. 4, 591–594 (2015).
10. Ruelle David & Floris Takens, On the nature of turbulence, Communications in Mathematical Physics, no. 20, 167-192 (1971).
11. E, Zeraoulia,"A new chaotic attractor from 2-D discrete mapping via border-collision period doubling scenario," Discrete dynamics in nature and society, Vol. 5, 235-238 (2005).
12. M. Mammeri, Symmetry and Periodic-Chaos in 3-D Sinusoid Discrete Maps, Bulletin of Mathematical Analysis and Applications, Vol. 9.no. 1, 1-9 (2017).
13. Lai, D and Chen, G. Making a discrete dynamical system chaotic:Theorical results and numerical simulations, International Journal of Bifurcation and Chaos, 13 (11), 3437-3442 (2003).
14. Nering, Evar D, Linear Algebra and Matrix Theory (2nd ed.), New York (1970).
15. Bronson, Richard, Matrix Methods: An Introduction, New York (1970).
16. Elaydi SN. An introduction to difference equations. New York: Springer, (1995).