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A Study of Unique Freedom System of Differential Transformation Method (DTM) For Numerical Simulations Solving Models

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ABSTRACT

In this paper, the differential transformation method (DTM) is employed to find the semi-analytical solutions of SIS and SI epidemic models for constant population. Firstly, the theoretical background of DTM is studied and followed by constructing the solutions of SIS and SI epidemic models. Furthermore, the convergence analysis of DTM is proven by proposing two theorems. Finally, numerical computations are made and compared with the exact solutions. From the numerical results, the solutions produced by DTM approach the exact solutions which agreed with the proposed theorems. It can be seen that the DTM is an alternative technique to be considered in solving many practical problems involving differential equations.

Keywords: Differential transformation method (DTM); exact solution; semi-analytical solution; SIS model; SI model

INTRODUCTION

There are many methods to solve differential equations. One of them is the Taylor series. The Taylor series, however, requires huge effort in order to find the derivatives of function. Moreover, it is very complicated to find the higher order derivatives of function. Due to these reasons, Zhou (1986) had proposed a new form of the Taylor series called the differential transformation method (DTM) and applied it to

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solve mathematical problems in electrical circuit analysis. The idea of the DTM is to determine the coefficients of the Taylor series of a function by solving the induced recursive equation from the given differential equation. The emergence of DTM has motivated many researchers to solve different types of differential equation. Chen and Ho (1996) had used it to construct the solution of partial differential equations while Jang and Chen (1997) had used the DTM to solve initial and boundary value problems. Later, Chen and Liu (1998) had employed this method to find the solution of two point boundary value problems. Next, Ayaz (2004) had constructed the solution of system of differential equations using DTM. In 2005, Abbasov and Bahadir (2005) had obtained semi-analytical solutions of linear and non-linear problems in engineering using the DTM. Hassan and Ertürk (2007) had used DTM to solve an elliptic partial differential equation. Later, Hassan (2008) had used this method to solve linear and non-linear system of differential equations. Due to its popularity in solving various types of equation, many authors had used the DTM to solve difference equations (Arikoglu & Ozkol 2006), fractional differential equations (Arikoglu & Ozkol 2007; Momani et al. 2008), volterra integral equations (Odibat 2008; Tari et al. 2009), integro-differential equations of fractional order (Nazari & Shahmorad 2010), Burgers and Schrödinger equations (Abazari & Borhanifar 2010; Borhanifar & Abazari 2011), fractional chaotic dynamical systems (Alomari 2011) and partial differential equations of order four (Soltanalizadeh & Branch 2012). These contributions showed that the DTM is widely used to solve many types of differential equation as stated. In finding the solutions of SIS and SIR epidemic models (Kermack & McKendrick 1927), many studies have been attempted. Nucci and Leach (2004) had used Lie group to present the explicit solution of SIS epidemic model while Khan et al. (2009) had solved SIS and SIR epidemic models by means of homotopy analysis method (HAM). Later, Shabir et al. (2010) had proposed exact solutions of SIR and SIS epidemic models. In 2013, Abubakar et al. had obtained approximate solution of SIR model using homotopy perturbation method (HPM). Many efforts havebeen given to solve the SIS and SIR epidemic models by several researchers (Jing & Zhu 2005; Korobeinikov & Wake 2002; Pietro 2007; Yicang & Liu 2003; Zhien et al. 2003). Certain epidemiology models have been solved by DTM (Akinboro et al. 2014; Batiha & Batiha 2011). Batiha and Batiha (2011) considered the numerical solution of SIR model without vital dynamics using DTM, meanwhile Akinboro et al. (2014) considered the numerical solution of SIR model with vital dynamics using DTM. However, to the best of our knowledge, SIS model without vital dynamics and SI model with vital dynamics are not solved by DTM yet. Therefore, this paper focused on finding semi-analytical solutions of the SIS model without vital dynamics and SI model with vital dynamics using DTM.

BASIC DEFINITIONS

The DTM is developed based on the Taylor series expansion. This method constructs an analytical or semi-analytical solution in the

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form of polynomial. The following basic definitions and fundamental properties are adopted from Hasan (2008).

Definition 1

A Taylor polynomial of degree is defined as follows:

$$p_n(x) = \sum_{k=0}^{n} \frac{1}{k!}\left(f^{(k)}(c)\right)(x-c)^k. \tag{1}$$

Theorem 2.1 Suppose that the function f has (n+1) derivatives on the interval (c – r, c + r), for some r > 0 and , for all where Rn (x) is the error between pn (x) and the approximated function f(x), then the Taylor series expanded about x = c converges to f(x) that is:

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(f^{(k)}(c)\right)(x-c)^k, \tag{2}$$

for all $x \in (c-r, c+r)$.

Definition 2 The differential transformation of the function f(x) for the k-th derivative is defined as follows:

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0}, \tag{3}$$

where f(x) is the original function and F(k) is the transformed function.

Definition 3 The inverse differential transformation of F(k) is defined as follows:

$$f(x) = \sum_{k=0}^{\infty} (x-x_0)^k F(k). \tag{4}$$

Substituting (3) into (4) yields:

$$f(x) = \sum_{k=0}^{\infty} (x-x_0)^k \frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0}. \tag{5}$$

Note that, this is the Taylor series of f(x) at x = x0 . The basic operations of DTM can be deduced from (4) and (5) as listed in Table 1.

SOLUTION OF SIS MODEL WITHOUT VITAL DYNAMICS USING DTM

SIS model without vital dynamics returns the infective to the susceptible class on recovery because the diseases confer no immunity against reinfection. The SIS model without vital dynamics is as follows (Shabbir et al. 2010):

$$\begin{cases} s'(t) = -rs(t)i(t) + \alpha i(t), \\ i'(t) = rs(t)i(t) - \alpha i(t), \end{cases} \tag{6}$$

subject to initial conditions:

$$i(0) = I_0, \qquad s(0) = S_0, \tag{7}$$

where s is the susceptible fraction of the population; i is the infected fraction of the population; r is the infectivity coefficient; and α is the recovery coefficient, while I0 > 0, r > 0, α > 0, S0 > 0.

By applying the DTM to (6), we obtained the following recurrence relations:

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$$S(k+1)=\frac{1}{k+1}\left[\left(-r\sum_{m=0}^{k} I(m)S(k-m)\right)+\alpha I(k)\right] \qquad (8)$$

$$I(k+1)=\frac{1}{k+1}\left[\left(r\sum_{m=0}^{k} I(m)S(k-m)\right)-\alpha I(k)\right] \qquad (9)$$

From (8) and (9) with initial conditions (7), we have:

$$S(0)=S_0,\ I(0)=I_0.$$

$k=0$,

$$S(1)=\alpha I_0-rI_0S_0,\ I(1)=-\alpha I_0+rI_0S_0$$

$k=1$,

TABLE 1. The fundamental operations of DTM

Original functions	Transformed functions
$y(x)=u(x)\pm m(x)$	$Y(k)=U(k)\pm M(k)$
$y(x)=\alpha m(x)$	$Y(k)=\alpha M(k)$
$y(x)=\frac{du(x)}{dx}$	$Y(k)=(k+1)\,U(k+1)$
$y(x)=\frac{d^2u(x)}{dx^2}$	$Y(k)=(k+1)(k+2)\,U(k+2)$
$y(x)=\frac{d^nu(x)}{dx^n}$	$Y(k)=(k+1)(k+2)\,\mathrm{K}\,(k+n)\,U(k+n)$
$y(x)=1$	$Y(k)=\delta(k)$
$y(x)=x$	$Y(k)=\delta(k-1)$
$y(x)=x^m$	$Y(k)=\delta(k-m)=\begin{cases}1, & k=m\\ 0, & k\neq m\end{cases}$
$y(x)=g(x)\,h(x)$	$\sum_{m=0}^{k} H(m)G(k-m)$
$y(x)=e^{\lambda x}$	$Y(k)=\frac{\lambda^k}{k!}$
$y(x)=(1+x)^m$	$Y(k)=\frac{m(m-1)\dots(m-k+1)}{k!}$

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$$S(2)=\frac{1}{2}\left(\alpha\left(-\alpha I_0+rI_0S_0\right)-r\left(I_0\left(\alpha I_0-rI_0S_0\right)+S_0\left(-\alpha I_0+rI_0S_0\right)\right)\right)$$

$$I(2)=\frac{1}{2}\left(-\alpha\left(-\alpha I_0+rI_0S_0\right)+r\left(I_0\left(\alpha I_0-rI_0S_0\right)+S_0\left(-\alpha I_0+rI_0S_0\right)\right)\right).$$

$k=2,$

$$\begin{aligned}S(3)=\frac{1}{3}(&\frac{1}{2}\alpha\left(-\alpha\left(-\alpha I_0+rI_0S_0\right)+r\left(I_0\left(\alpha I_0-rI_0S_0\right)+S_0\left(-\alpha I_0+rI_0S_0\right)\right)\right)-r\left(\left(\alpha I_0-rI_0S_0\right)\right.\\&\left(-\alpha I_0+rI_0S_0\right)+\frac{1}{2}I_0\left(\alpha\left(-\alpha I_0+rI_0S_0\right)-r\left(I_0\left(\alpha I_0-rI_0S_0\right)+S_0\left(-\alpha I_0+rI_0S_0\right)\right)\right)+\\&\left.\frac{1}{2}S_0\left(-\alpha\left(-\alpha I_0+rI_0S_0\right)+r\left(I_0\left(\alpha I_0-rI_0S_0\right)+S_0\left(-\alpha I_0+rI_0S_0\right)\right)\right)\right)).\end{aligned}$$

$$\begin{aligned}I(3)=\frac{1}{3}(&-\frac{1}{2}\alpha\left(-\alpha\left(-\alpha I_0+rI_0S_0\right)+r\left(I_0\left(\alpha I_0-rI_0S_0\right)+S_0\left(-\alpha I_0+rI_0S_0\right)\right)\right)+r\left(\left(\alpha I_0-rI_0S_0\right)\right.\\&\left(-\alpha I_0+rI_0S_0\right)+\frac{1}{2}I_0\left(\alpha\left(-\alpha I_0+rI_0S_0\right)-r\left(I_0\left(\alpha I_0-rI_0S_0\right)+S_0\left(-\alpha I_0+rI_0S_0\right)\right)\right)+\\&\left.\frac{1}{2}S_0\left(-\alpha\left(-\alpha I_0+rI_0S_0\right)+r\left(I_0\left(\alpha I_0-rI_0S_0\right)+S_0\left(-\alpha I_0+rI_0S_0\right)\right)\right)\right)).\end{aligned}$$

$$\vdots$$

We define the general solution of SIS model as follows:

$$s(t)=\sum_{k=0}^{\infty}S(k)t^k, \tag{10}$$

$$i(t)=\sum_{k=0}^{\infty}I(k)t^k. \tag{11}$$

In this manner, s(t) and i(t) for $k \geq 2$ can be easily obtained. Therefore, from (10) and (11), the first four terms of the series solutions.

CONCLUSION

In this paper, we solved epidemic models of SIS without vital dynamics and SI with vital dynamics by the differential transformation method (DTM). Numerical experimentationsshowed that the approximate solutions have excellent accuracy and higher accuracy can be achieved by increasing the order of the DTM. For future work, we expect that DTM could be extended to solve epidemiology models in fractional order and also other epidemiology models with various compartment designs.

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