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Convergence Analysis of Triangular and Symmetric Splitting Method for Steady State Vector of Regularized Linear System with Block Circulant Matrices

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Abstract

In this paper, triangular and symmetric splitting method is applied for determining the steady state probability vector π of regularized linear system $Ax = b.$ The homogeneous system $\pi \Re = 0,$ where $\Re$ is block stochastic rate matrix, is transformed into regularized non-homogeneous linear system $Ax = b$ by using preconditioned matrix with the small perturbation $\varepsilon > 0.$ It is proved that the regularized matrix $A = \Re^T + \varepsilon I$ is positive definite. In this convergence analysis of the (triangular and symmetric) TS iteration method, both relative error and absolute error are considered. From the numerical results, it is concluded that TS iteration method converge rapidly when compared with other existing methods.

Keywords: *Block circulant matrix, Steady state probability vector, Stochastic rate matrix, TS Method, Preconditioned matrix*

1. Introduction

Many problems in Engineering & Science give rise to homogeneous system of linear equations in the following form [1-5]:

$$\pi\Re = 0 \quad \text{and} \quad \pi e = 1, \qquad \dots (1)$$

where

$$\Re = \begin{bmatrix} \Re_1 & \Re_2 & \Re_3 & \dots & \Re_n \\ \Re_n & \Re_1 & \Re_2 & \dots & \Re_{n-1} \\ \Re_{n-1} & \Re_n & \Re_1 & \cdots & \Re_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Re_2 & \Re_3 & \Re_4 & \cdots & \Re_1 \end{bmatrix}, \qquad \dots (2)$$

and

$$\Re_i = \begin{bmatrix} r_{i1} & r_{i2} & r_{i3} & r_{i4} & \cdots & r_{in} \\ r_{in} & r_{i1} & r_{i2} & r_{i3} & \cdots & r_{in-1} \\ r_{in-1} & r_{in} & r_{i1} & r_{i2} & \cdots & r_{in-2} \\ r_{in-2} & r_{in-1} & r_{in} & r_{i1} & \cdots & r_{in-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{i2} & r_{i3} & r_{i4} & r_{i5} & \cdots & r_{i1} \end{bmatrix}, \quad 1 \le j \le n.$$

If $\sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij} = 0,$ for $r_{11} > 0, r_{ij} < 0$, and $1 \le i \le n,$ then $\sum_{i=1}^{m} R_i = 0.$ Therefore, the matrix $\Re$ is doubly block circulant stochastic rate matrix, and $\Re_i$ is circulant matrix but not stochastic rate matrix.

Taking transpose on both sides of homogeneous Eq. (1),

$$\pi \Re = 0,$$

JOURNAL OF ALGEBRAIC STATISTICS
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$$\Rightarrow (\pi\, \Re)^T = 0,$$
$$\Rightarrow \Re^T\, \pi^T = 0,$$
$$\Rightarrow \overline{A}\, x = 0\,, \qquad \ldots (3)$$

where $\overline{A} = \Re^T$ and $x = \pi^T$, the row and column sum of the coefficient matrix $\overline{A}$ are zero, diagonal elements are positive, and off-diagonal elements are non-positive.

The homogeneous system (3) gives a unique one dimensional null solution (or) infinitely many solutions. If the system (3) possesses infinite number of solutions, then the real time physical system pertains to the Eq.(3) is not stable. If the system (3) possesses unique non-zero solution, then the real time system (3) is stable. For the unique non- zero solution, the preconditioned matrix with small perturbation $\varepsilon > 0$ is constant and the homogeneous system (3) is transformed to non-homogeneous system [6-7],

$$Ax = (Q^T + \varepsilon I)x = e_{n^2} = b \qquad \ldots (4)$$

where $e_{n^2} = b$ is a unit vector given by $e_{n^2} = [0, 0, \ldots, 0, 1]^T$. The steady-state probability distribution vector π is then obtained by normalizing the vector x.

The steady state vector π is computed by many researchers with direct and iterative methods [19]. The significant improvement in convergence rates can be achieved through Krylov subspace methods [8], preconditioning techniques [9-11], and two splitting and multi splitting iterative methods [12-15]. In the paper [17], the steady state probability vector of positive definite linear systems of block circulant stochastic probability matrix is obtained by triangular and skew-symmetric (TSS) iteration method. In this paper, Triangular and Symmetric splitting iteration method is employed for class of block circulant stochastic rate matrix. In which, the matrix $\Re$ is doubly block circulant stochastic rate matrix and the block sub matrices are circulant matrices but not stochastic rate matrices. Hence, in this paper, an improved convergence solution is developed for the regularized positive definite linear system of block circulant stochastic rate matrix. Moreover, the contraction factor α and convergence criteria of the regularized linear system using inexact triangular symmetric (ITS) splitting method are obtained. The organization of the paper is as follows.

In section 2, Basic definitions and conditions for the convergence analysis of solution of regularized linear system are discussed. In section 3, TS iteration procedure and its convergence is discussed. In section 4, the choice of contraction factor α is discussed. Numerical results are preserved in section 5. Finally conclusions are given in section 6.

2. Conditions for convergence analysis

In this section, some basic definitions are given, which are useful to prove regularized matrix is positive definite and then, we prove some theorems for the unique convergence solution of the regularized linear system.

Definition 1: Any matrix $A \in R^{n^2 \times n^2}$ of the form $A = sI - P,\ s > 0,\ P \geq 0$ is called an *M*-matrix if $s \geq \rho(P)$. If $s > \rho(P)$ then A is non-singular *M*-matrix otherwise, A is singular *M*-matrix.

Definition 2: A non symmetric matrix *A* is positive definite, if its symmetric part i.e., $\left(\dfrac{A + A^T}{2}\right)$ is positive definite.

JOURNAL OF ALGEBRAIC STATISTICS
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Theorem 1: If $\Re \in R^{n^2 x n^2}$ is a circulant stochastic rate matrix with block circulant matrices, then there exist $\varepsilon > 0$ such that, $A = \Re^T + \varepsilon\, I_{n^2}$ is positive definite.

Proof: For proving, A is positive definite, it is enough to prove that its symmetric part i.e. $\frac{A + A^T}{2}$ is positive definite.

We have

$$\Re = \begin{bmatrix} \Re_1 & \Re_2 & \Re_3 & \Re_4 & \cdots & \Re_n \\ \Re_n & \Re_1 & \Re_2 & \Re_3 & \cdots & \Re_{n-1} \\ \Re_{n-1} & \Re_n & \Re_1 & \Re_2 & \cdots & \Re_{n-2} \\ \Re_{n-2} & \Re_{n-1} & \Re_n & \Re_1 & \cdots & \Re_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \Re_2 & \Re_3 & \Re_4 & \Re_5 & \cdots & \Re_1 \end{bmatrix} \text{ and } \Re^T = \begin{bmatrix} \Re_1 & \Re_n & \Re_{n-1} & \Re_{n-2} & \cdots & \Re_2 \\ \Re_2 & \Re_1 & \Re_n & \Re_{n-1} & \cdots & \Re_3 \\ \Re_3 & \Re_2 & \Re_1 & \Re_n & \cdots & \Re_4 \\ \Re_4 & \Re_3 & \Re_2 & \Re_1 & \cdots & \Re_5 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \Re_n & \Re_{n-1} & \Re_{n-2} & \Re_{n-3} & \cdots & \Re_1 \end{bmatrix}$$

$$\Rightarrow \frac{\Re + \Re^T}{2} = \begin{bmatrix} \Re_1 & \frac{\Re_2 + \Re_n}{2} & \frac{\Re_3 + \Re_{n-1}}{2} & \frac{\Re_4 + \Re_{n-2}}{2} & \cdots & \frac{\Re_2 + \Re_n}{2} \\ \frac{\Re_2 + \Re_n}{2} & \Re_1 & \frac{\Re_2 + \Re_n}{2} & \frac{\Re_3 + \Re_{n-1}}{2} & \cdots & \frac{\Re_3 + \Re_{n-1}}{2} \\ \frac{\Re_3 + \Re_{n-1}}{2} & \frac{\Re_2 + \Re_n}{2} & \Re_1 & \frac{\Re_2 + \Re_n}{2} & \cdots & \frac{\Re_4 + \Re_{n-2}}{2} \\ \frac{\Re_4 + \Re_{n-2}}{2} & \frac{\Re_3 + \Re_{n-1}}{2} & \frac{\Re_2 + \Re_n}{2} & \Re_1 & \cdots & \frac{\Re_5 + \Re_{n-3}}{2} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{\Re_2 + \Re_n}{2} & \frac{\Re_3 + \Re_{n-1}}{2} & \frac{\Re_4 + \Re_{n-2}}{2} & \frac{\Re_5 + \Re_{n-3}}{2} & \cdots & \Re_1 \end{bmatrix}$$

We have, $A = \Re^T + \varepsilon I_{n^2}$

$$\Rightarrow \frac{A + A^T}{2} = \frac{\Re + \Re^T}{2} + \varepsilon I_{n^2}$$

$$= \begin{bmatrix} \Re_1 + \varepsilon I_n & \frac{\Re_2 + \Re_n}{2} & \frac{\Re_3 + \Re_{n-1}}{2} & \frac{\Re_4 + \Re_{n-2}}{2} & \cdots & \frac{\Re_2 + \Re_n}{2} \\ \frac{\Re_2 + \Re_n}{2} & \Re_1 + \varepsilon I_n & \frac{\Re_2 + \Re_n}{2} & \frac{\Re_3 + \Re_{n-1}}{2} & \cdots & \frac{\Re_3 + \Re_{n-1}}{2} \\ \frac{\Re_3 + \Re_{n-1}}{2} & \frac{\Re_2 + \Re_n}{2} & \Re_1 + \varepsilon I_n & \frac{\Re_2 + \Re_n}{2} & \cdots & \frac{\Re_4 + \Re_{n-2}}{2} \\ \frac{\Re_4 + \Re_{n-2}}{2} & \frac{\Re_3 + \Re_{n-1}}{2} & \frac{\Re_2 + \Re_n}{2} & \Re_1 + \varepsilon I_n & \cdots & \frac{\Re_5 + \Re_{n-3}}{2} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{\Re_2 + \Re_n}{2} & \frac{\Re_3 + \Re_{n-1}}{2} & \frac{\Re_4 + \Re_{n-2}}{2} & \frac{\Re_5 + \Re_{n-3}}{2} & \cdots & \Re_1 + \varepsilon I_n \end{bmatrix} = (r_{11} + \varepsilon) I_{n^2} - P$$

where

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$$P=\begin{bmatrix} \Re_1^1 & \frac{\Re_2+\Re_n}{2} & \frac{\Re_3+\Re_{n-1}}{2} & \frac{\Re_4+\Re_{n-2}}{2} & \cdots & \frac{\Re_2+\Re_n}{2} \\ \frac{\Re_2+\Re_n}{2} & \Re_1^1 & \frac{\Re_2+\Re_n}{2} & \frac{\Re_3+\Re_{n-1}}{2} & \cdots & \frac{\Re_3+\Re_{n-1}}{2} \\ \frac{\Re_3+\Re_{n-1}}{2} & \frac{\Re_2+\Re_n}{2} & \Re_1^1 & \frac{\Re_2+\Re_n}{2} & \cdots & \frac{\Re_4+\Re_{n-2}}{2} \\ \frac{\Re_4+\Re_{n-2}}{2} & \frac{\Re_3+\Re_{n-1}}{2} & \frac{\Re_2+\Re_n}{2} & \Re_1^1 & \cdots & \frac{\Re_5+\Re_{n-3}}{2} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{\Re_2+\Re_n}{2} & \frac{\Re_3+\Re_{n-1}}{2} & \frac{\Re_4+\Re_{n-2}}{2} & \frac{\Re_5+\Re_{n-3}}{2} & \cdots & \Re_1^1 \end{bmatrix} \geq 0, \quad \text{and } \Re_1^1=(r_{11}+\varepsilon)I_n-\Re_1$$

If P is non-negative symmetric matrix, then we have to find the spectral radius of the matrix P. From the papers [14, 17], one can have

$$\rho(P)=r_{12}+r_{13}+\ldots+r_{1n}+r_{21}+r_{22}+\ldots+r_{2n}+\ldots+r_{n1}+r_{n2}+\ldots+r_{nn}$$
$$=r_{11},$$
$$\Rightarrow r_{11}+\varepsilon>r_{11}, \quad \text{for } \varepsilon>0,$$
$$\Rightarrow r_{11}+\varepsilon>r_{11}=\rho(P),$$
$$\Rightarrow A \text{ is a positive definite matrix.}$$

In next section, the TS iteration method applied for finding the unique non-zero solution of the non-homogeneous regularized positive-definite linear system $Ax=b$ as given in (4). It is clear that, solving the linear system (4) may tend to a small perturbation error of O(ϵ), but Ching et al. have proved that the 2-norm of the error introduced by the regularization tends to zero and error analysis depicts in the numerical results. The proof for this regularization technique and convergence analysis can be found in the [6, 12-15].

3. Triangular and Symmetric Splitting Iterative Method

In this section, first TS splitting method on regularized linear system (4) of block circulant transition rate matrix for finding the steady state probability vector π is applied,and then analyzed for its convergence criteria. Consider the coefficient matrix *A* of the regularized linear system (4) that can splitted into the following form,

$$A=(L+D-U^T)+(U+U^T)=T+S$$

where
$$L=\begin{bmatrix} \frac{1}{3}\varepsilon I_n & 0 & 0 & \cdots & 0 \\ -\Re_2 & \frac{1}{3}\varepsilon I_n & 0 & \cdots & 0 \\ -\Re_3 & -\Re_2 & \frac{1}{3}\varepsilon I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -\Re_n & -\Re_{n-1} & -\Re_{n-2} & \cdots & \frac{1}{3}\varepsilon I_n \end{bmatrix}, U=\begin{bmatrix} \left(\frac{r_{11}}{2}+\frac{\varepsilon}{3}\right)I_n & -\Re_n & -\Re_{n-1} & \cdots & -\Re_2 \\ 0 & \left(\frac{r_{11}}{2}+\frac{\varepsilon}{3}\right)I_n & -\Re_n & \cdots & -\Re_3 \\ 0 & 0 & \left(\frac{r_{11}}{2}+\frac{\varepsilon}{3}\right)I_n & \cdots & -\Re_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \left(\frac{r_{11}}{2}+\frac{\varepsilon}{3}\right)I_n \end{bmatrix},$$

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 03, 2022, p. 4585-4596
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and $$D=\begin{bmatrix} \left(\frac{r_{11}}{2}+\frac{\varepsilon}{3}\right)I_n & 0 & 0 & 0 & 0 \\ 0 & \left(\frac{r_{11}}{2}+\frac{\varepsilon}{3}\right)I_n & 0 & 0 & 0 \\ 0 & 0 & \left(\frac{r_{11}}{2}+\frac{\varepsilon}{3}\right)I_n & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \left(\frac{r_{11}}{2}+\frac{\varepsilon}{3}\right)I_n \end{bmatrix}$$

are the lower, upper, and diagonal block matrices respectively and $(L+D-U^T)=T$ is a block triangular matrix, and $S=U+U^T$ block symmetric matrix. Given an initial guess $x^{(0)}$, compute $x^{(k)}$ *for* $k=1,2,3.\cdots$ by using

$$(\alpha I_{n^2}+T)x^{(k+1/2)}=(\alpha I_{n^2}-S)x^{(k)}+b$$

$$(\alpha I_{n^2}+S)x^{(k+1)}=(\alpha I_{n^2}-T)x^{(k+1/2)}+b \qquad \text{... (5)}$$

until $\{x^{(k)}\}$ converges for any positive constant α.

Without loss of generality, Eq. (5) can be written as

$$x^{(k+1)}=M(\alpha)x^{(k)}+N(\alpha)b \qquad \text{... (6)}$$

where

$$M(\alpha)=\left(\alpha I_{n^2}+S\right)^{-1}\left(\alpha I_{n^2}-T\right)\left(\alpha I_{n^2}+T\right)^{-1}\left(\alpha I_{n^2}-S\right)$$

$$\textit{and} \qquad N(\alpha)=2\alpha\left(\alpha I_{n^2}+S\right)^{-1}\left(\alpha I_{n^2}+T\right)^{-1}$$

Here, $M(\alpha)$ is called iteration matrix of TS iteration method (5). The solution of TS iteration method converges if $\rho(M(\alpha))<1$.

Lemma 2. If *S* is the real block symmetric matrix, then S is positive definite.

Proof: We have $$U=\begin{bmatrix} \left(\frac{r_{11}}{2}+\frac{\varepsilon}{3}\right)I_n & -\Re_n & -\Re_{n-1} & \cdots & -\Re_2 \\ 0 & \left(\frac{r_{11}}{2}+\frac{\varepsilon}{3}\right)I_n & -\Re_n & \cdots & -\Re_3 \\ 0 & 0 & \left(\frac{r_{11}}{2}+\frac{\varepsilon}{3}\right)I_n & \cdots & -\Re_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \left(\frac{r_{11}}{2}+\frac{\varepsilon}{3}\right)I_n \end{bmatrix}$$ then

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 03, 2022, p. 4585-4596
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$$U^T = \begin{bmatrix} \left(\frac{r_{11}}{2}+\frac{\varepsilon}{3}\right)I_n & 0 & 0 & \cdots & 0 \\ -\Re_n & \left(\frac{r_{11}}{2}+\frac{\varepsilon}{3}\right)I_n & 0 & \cdots & 0 \\ -\Re_{n-1} & -\Re_n & \left(\frac{r_{11}}{2}+\frac{\varepsilon}{3}\right)I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\Re_2 & -\Re_3 & -\Re_4 & \cdots & \left(\frac{r_{11}}{2}+\frac{\varepsilon}{3}\right)I_n \end{bmatrix}$$

$$\Rightarrow S = U + U^T = \begin{bmatrix} \left(r_{11}+\frac{2\varepsilon}{3}\right)I_n & -\Re_n & -\Re_{n-1} & \cdots & -\Re_2 \\ -\Re_n & \left(r_{11}+\frac{2\varepsilon}{3}\right)I_n & -\Re_n & \cdots & -\Re_3 \\ -\Re_{n-1} & -\Re_n & \left(r_{11}+\frac{2\varepsilon}{3}\right)I_n & \cdots & -\Re_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\Re_2 & -\Re_3 & -\Re_4 & \cdots & \left(r_{11}+\frac{2\varepsilon}{3}\right)I_n \end{bmatrix}$$

$$= \left(r_{11}+\frac{2\varepsilon}{3}\right)I_n - B$$

where, $$B = \begin{bmatrix} 0 & -\Re_n & -\Re_{n-1} & \cdots & -\Re_2 \\ -\Re_n & 0 & -\Re_n & \cdots & -\Re_3 \\ -\Re_{n-1} & -\Re_n & 0 & \cdots & -\Re_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\Re_2 & -\Re_3 & -\Re_4 & \cdots & 0 \end{bmatrix}$$

$$\therefore \rho(B) = r_{12} + r_{13} + r_{14} + \dots\dots\dots + r_{1n} = r_{11}$$

$$\Rightarrow \left(r_{11}+\frac{2\varepsilon}{3}\right) > \rho(B)$$

Therefore, S is positive definite.

Lemma 3 [15]. Let $\Im(\alpha) = \left(\alpha I_{n^2} + S\right)^{-1}\left(\alpha I_{n^2} - S\right)$. If $S \in R^{n^2\times n^2}$ is a positive-definite matrix, then $\|\Im(\alpha)\|_2 < 1, \forall \alpha > 0.$

Theorem 3: If $A \in R^{nxn}$ is a block circulant matrix for a regularized linear system (4), and $M(\alpha)$ is a block matrix of TS iteration method, then $\rho\left(M(\alpha)\right) < 1$.

Proof: From (6), we have $M(\alpha) = \left(\alpha I_{n^2} + S\right)^{-1}\left(\alpha I_{n^2} - T\right)\left(\alpha I_{n^2} + T\right)^{-1}\left(\alpha I_{n^2} - S\right)$

$$\Rightarrow \quad \rho\left(M(\alpha)\right) = \left\|\left(M(\alpha)\right)\right\|_2 = \left\|\left(\alpha I_{n^2} + S\right)^{-1}\left(\alpha I_{n^2} - T\right)\left(\alpha I_{n^2} + T\right)^{-1}\left(\alpha I_{n^2} - S\right)\right\|_2$$

$$\le \left\|\left(\alpha I_{n^2} - T\right)\left(\alpha I_{n^2} + T\right)^{-1}\right\|_2 \left\|\left(\alpha I_{n^2} + S\right)^{-1}\left(\alpha I_{n^2} - S\right)\right\|_2 = \|\square(\alpha)\Im(\alpha)\|_2 = \|\square(\alpha)\|_2\,\|\Im(\alpha)\|_2 \ \dots (7)$$

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 03, 2022, p. 4585-4596
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From Eqs. (7) & Lemma 3, $\rho\left(M\left(\alpha\right)\right) = \left\| \left(M\left(\alpha\right)\right) \right\|_2 \leq \left\| \left(\alpha I_{n^2} - T\right)\left(\alpha I_{n^2} + T\right)^{-1} \right\|_2$ ... (8)

Let $\square\left(\alpha\right) = \left(\alpha I_{n^2} - T\right)\left(\alpha I_{n^2} + T\right)^{-1}$. Since T is a triangular matrix and which is positive definite [15], then $\left\|\square\left(\alpha\right)\right\|_2 < 1, \quad \forall\, \alpha > 0$ … (9)

From Eqs. (8) & (9),

$$\rho\left(M\left(\alpha\right)\right) = \left\|M\left(\alpha\right)\right\|_2 \leq \left\|\Im\left(\alpha\right)\right\| \left\|\square\left(\alpha\right)\right\| < 1$$

$$\Rightarrow \quad \rho\left(M\left(\alpha\right)\right) < 1, \quad \forall\, \alpha > 0.$$

Therefore, the iterative solution of the regularized linear system converges to a unique solution.

4. Estimation of the contraction factor α and Inner Iteration Method

In this section, the choice of contraction factor α is discussed and convergence criteria for the iterative solution of the regularized linear system. The following investigation describes formulae in approximating the contraction factor α by using circulant transition rate matrix with block circulant sub matrices. Contraction factor α plays an important role in finding the convergence solution for the given linear system. The parameter α is minimizes the upper bound of $\rho(M(\alpha))$. The block coefficient matrix A of the regularized linear system Eq. (4) can be splitted by block triangular matrix and block skew symmetric matrix [5],

i.e.,
$$\begin{aligned} A &= (L + D - U^T) + (U + U^T) \\ &= T_1 + S_1 \\ &= (U + D - L^T) + (L + L^T) \\ &= T_2 + S_2 \end{aligned}$$

where *D* is a block diagonal matrix, *L* and *U* are strictly block lower and upper triangular matrices of block coefficient matrix *A* of a regularized linear system (4) .

Let $H_1 = U^T - L$ and $H_2 = L^T - U$ be strictly lower and upper block triangular matrices.

Now, $\left(\alpha I_{n^2} + T_i\right)^{-1} = \left(\alpha I_{n^2} + D + L - U^T\right)^{-1} = \left(\alpha I_{n^2} + D + L^T - U\right)^{-1}$ for i=1, 2

$$\begin{aligned} \left(\alpha I_{n^2} + T_i\right)^{-1} &= \left(\alpha I_{n^2} + D - (U^T - L\right)^{-1} = \left(\alpha I_{n^2} + D - (U - L^T)\right)^{-1} \\ &= \left(\left(\alpha I_{n^2} + D\right) - H_i\right)^{-1}, \text{ for } i=1, 2 \\ &= \left(\alpha I_{n^2} + D\right)^{-1}\left(I_{n^2} - \left(\alpha I_{n^2} + D\right)^{-1} H_i\right)^{-1} \\ &= \left(\alpha I_{n^2} + D\right)^{-1}\left(I_{n^2} - \left(\alpha I_{n^2} + D\right)^{-1} H_i\right)^{-1} \qquad .... (10) \end{aligned}$$

Since $\left(\alpha I_{n^2} + D\right)^{-1}$ is a diagonal matrix and H_i is a triangular matrix then $\left(\left(\alpha I_{n^2} + D\right)^{-1} H_1\right)$ is a triangular matrix. If $\rho\left(\left(\alpha I_{n^2} + D\right)^{-1} H_i\right) < 1$ then the infinite series $\left(I_{n^2} - \left(\alpha I_{n^2} + D\right)^{-1} H_i\right)^{-1}$ is convergent.

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 03, 2022, p. 4585-4596
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$$\therefore \quad \left(\alpha I_{n^2} + T_i\right)^{-1} = \left(\alpha I_{n^2} + D\right)^{-1} \left(I_{n^2} + \left(\alpha I_{n^2} + D\right)^{-1} H_i + \left(\left(\alpha I_{n^2} + D\right)^{-1}\right)^2 H_i^2 + \cdots\cdots \right)$$

(neglect the higher order approximations)

$$\left(\alpha I_{n^2} + T_i\right)^{-1} = \left(\alpha I_{n^2} + D\right)^{-1} \left(I_{n^2} + H_i \left(\alpha I_{n^2} + D\right)^{-1} \right)$$

$$\Rightarrow \left(\alpha I_{n^2} - T_i\right)\left(\alpha I_{n^2} + T_i\right)^{-1} = \left(\alpha I_{n^2} - D + H_i\right)\left(\alpha I_{n^2} + D\right)^{-1} \left(I_{n^2} + H_i \left(\alpha I_{n^2} + D\right)^{-1} \right)$$

$$\Rightarrow \left\|\left(\alpha I_{n^2} - T_i\right)\left(\alpha I_{n^2} + T_i\right)^{-1}\right\|_2 \approx \left\|\left(\alpha I_{n^2} - D\right) \left(\alpha I_{n^2} + D\right)^{-1}\right\|_2$$

$$\approx \left\|\left(\alpha I_n - D_j\right)\left(\alpha I_n + D_j\right)^{-1}\right\|_2$$ where D_j are the diagonal entries of the sub matrix $\Re_j$. The contraction factor α is obtained directly from the papers [12-17] as follows:

$$\left\|\left(\alpha I_{n^2} - T_i\right)\left(\alpha I_{n^2} + T_i\right)^{-1}\right\|_2 \approx \left\|\left(\alpha I_n - D_j\right) \left(\alpha I_n + D_j\right)^{-1}\right\|_2$$

$$\alpha^* \approx \arg \min_{\alpha > 0} \max \left|\frac{\alpha - d_{\min}}{\alpha - d_{\max}}\right|$$

$$\alpha^* = \sqrt{d_{\min}\ d_{\max}}$$

where $d_{\min}$ and $d_{\max}$ are minimum and maximum diagonal entries of the matrix D_j. Therefore the contraction factor $\alpha = \alpha^* = \frac{r_{11}}{2} + \frac{\varepsilon}{3}$ is minimizes the upper bound of the $\rho(M(\alpha))$. Computing the optimal parameter α is a hard task that needs in-depth study. The two half steps at each step of the TS iteration method requires finding the coefficient matrices $\left(\alpha\, I_{n^2} + T_i\right)$ and $\left(\alpha\, I_{n^2} + S_i\right)$ for $i = 1, 2$. To improve computing efficiency of the TS iteration, we apply ITS iteration, i.e. to solve the two sub problems iteratively. The first subsystems coefficient matrix $\left(\alpha\, I_{n^2} + T_i\right)$ and the second sub systems coefficient matrix $\left(\alpha\, I_{n^2} + S_i\right)$ is solved by Krylov subspace methods [8-9]. The first subsystems can be obtained [8-9], and the second subsystems, and coefficient matrix is obtained by CGS method [18].

5. Numerical Results

In this section**,** the error analysis of TSS, Jacobi, and TS methods by using block transition probability matrix with block circulant matrices are depicted for illustration purpose. Consider block circulant transition probability matrix as

$$Q = \begin{bmatrix} \begin{bmatrix} 0.825 & -0.15 \\ -0.15 & 0.825 \end{bmatrix} & \begin{bmatrix} -0.0125 & -0.2375 \\ -0.2375 & -0.0125 \end{bmatrix} & \begin{bmatrix} -0.025 & -0.15 \\ -0.15 & -0.025 \end{bmatrix} & \begin{bmatrix} -0.175 & -0.075 \\ -0.075 & -0.175 \end{bmatrix} \\ \begin{bmatrix} -0.175 & -0.075 \\ -0.075 & -0.175 \end{bmatrix} & \begin{bmatrix} 0.825 & -0.15 \\ -0.15 & 0.825 \end{bmatrix} & \begin{bmatrix} -0.0125 & -0.2375 \\ -0.2375 & -0.0125 \end{bmatrix} & \begin{bmatrix} -0.025 & -0.15 \\ -0.15 & -0.025 \end{bmatrix} \\ \begin{bmatrix} -0.025 & -0.15 \\ -0.15 & -0.025 \end{bmatrix} & \begin{bmatrix} -0.175 & -0.075 \\ -0.075 & -0.175 \end{bmatrix} & \begin{bmatrix} 0.825 & -0.15 \\ -0.15 & 0.825 \end{bmatrix} & \begin{bmatrix} -0.0125 & -0.2375 \\ -0.2375 & -0.0125 \end{bmatrix} \\ \begin{bmatrix} -0.0125 & -0.2375 \\ -0.2375 & -0.0125 \end{bmatrix} & \begin{bmatrix} -0.025 & -0.15 \\ -0.15 & -0.025 \end{bmatrix} & \begin{bmatrix} -0.175 & -0.075 \\ -0.075 & -0.175 \end{bmatrix} & \begin{bmatrix} 0.825 & -0.15 \\ -0.15 & 0.825 \end{bmatrix} \end{bmatrix}$$

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Consider the initial vector as $x^{(0)} = [0\,0\,0\,0\,0\,0\,0\,1]^T$, and then compute steady state vector of regularized linear system and corresponding relative error based on the values of $\varepsilon.$, contraction factor α . The stopping criteria is set as the relative error $<10^{-16}$. The results are depicted in Figs.1-3. Fig.1, depicts the convergence rate of iterative solution of regularized linear system with relative error for the particular value of contraction factor is $\alpha = 0.825$, and which numerically equivalent to the diagonal value of the matrix $\Re$, for different values of $\varepsilon.$ From this figure, it can be concluded that the relative error of iterative solution TS method converge rapidly as $\varepsilon.$ decreases. Fig.2, depicts the convergence rate of iterative solution of regularized linear system with relative error of Jacobi's, TSS, and TS methods for the particular value of contraction factor is $\alpha = 0.825$, $\varepsilon = 0.12$. From this figure, it can be concluded that, TS method converge rapidly than the Jacobi's and TSS iterative methods. Figs.3 illustrates the result for relative error and absolute error of Jacobi's, TSS, and TS methods in the case of contraction factor $\alpha = 0.825$, $\varepsilon = 0.12$. From this figure, it can be conclude the relative error converge faster than absolute error with accepting relative error in fig.2. Table 1, lists the optimal iteration parameters of the tested methods. For Jacobi, TSS and TS methods, the optimal parameters chosen based on Theorem 3. It is inferred observe that the changes in relative error and absolute errors are based on the different values of $\varepsilon.$. Therefore it is concluded that TS method converge rapidly when compared to the other existing Jacobi and TSS methods.

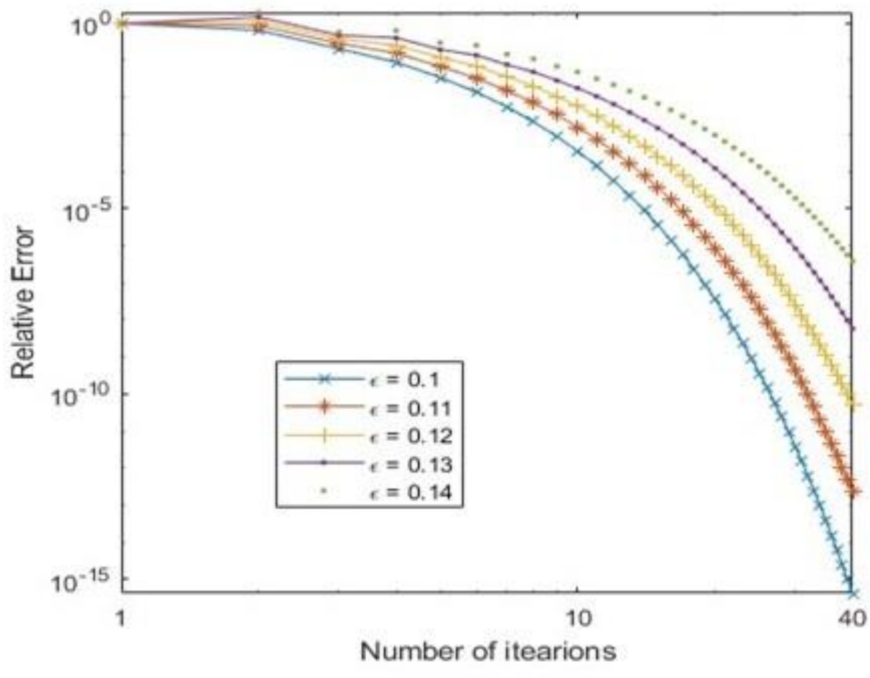

Fig. 1 Relative error of TS method for the contraction factor $\alpha = 0.825,$ and different values of ε .

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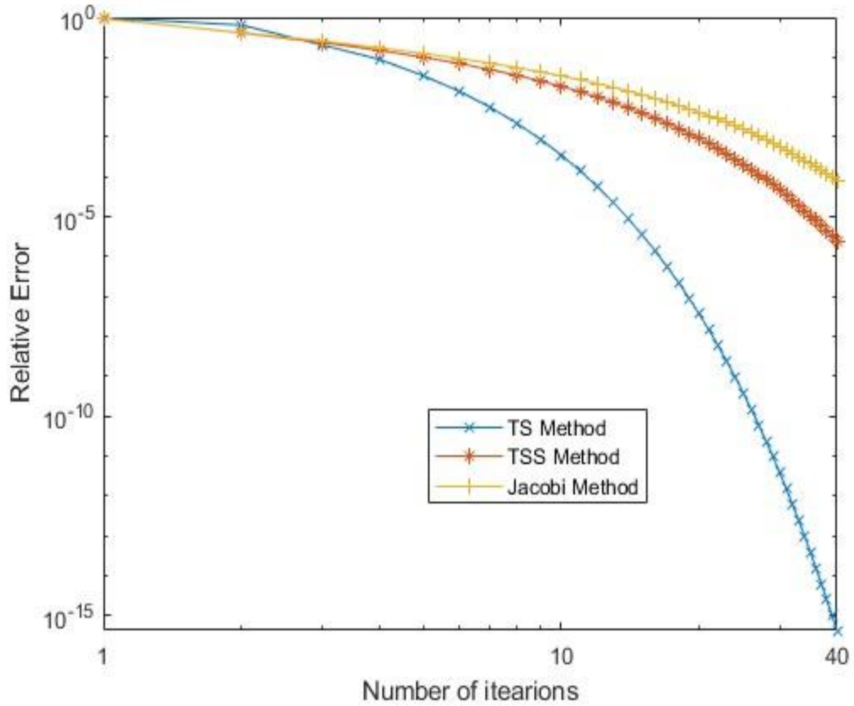

Fig.2 Relative error of TS, Jacobi method and TSS methods for the contraction factor $\alpha = 0.825,\ \varepsilon. = 0.12$.

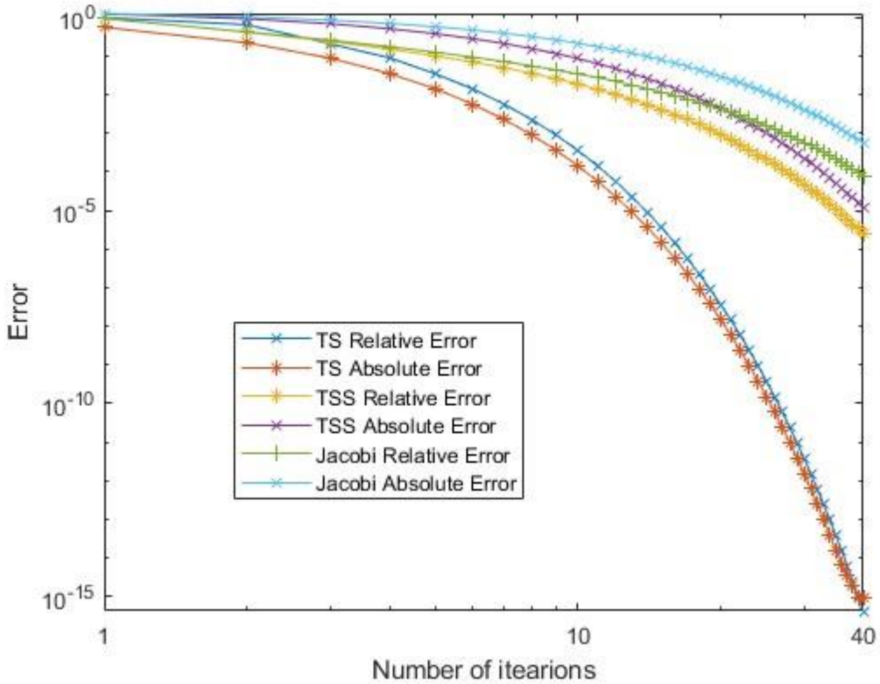

Fig.3 Relative error and Absolute error of TS, Jacobi method and TSS methods for the contraction factor $\alpha = 0.825$, $\varepsilon = 0.12$.

		TS Method		TSS Method		Jacobi Method	
	n	Relative Error	Absolute Error	Relative Error	Absolute Error	Relative Error	Absolute Error
$\varepsilon = 0.1$	1	0.034282	0.547489	0.099966	0.227905	0.123844	0.249319
	5	0.000352	0.013944	0.025412	0.374468	0.042932	0.567281
	10	8.971967e-06	0.000141	0.005451	0.084870	0.014245	0.211464
	15	9.128678e-08	1.443562e-06	0.001220	0.0192353	0.005101	0.078827
	20	9.288151e-10	1.468780e-08	0.000275	0.004359	0.001874	0.029384
	25	9.450410e-12	1.494439e-10	6.249362e-05	0.000988	0.000694	0.010953
	30	9.623018e-14	1.521734e-12	1.416177e-05	0.000223	0.000258	0.004083
	35	1.015655e-15	1.606105e-14	3.209567e-06	5.075397e-05	9.629405e-05	0.001522
	40	5.61658e-17	8.881784e-16	7.274211e-07	1.150303e-05	3.588547e-05	0.000567
$\varepsilon = 0.12$	1	0.120514	1.794552	0.091589	1.060650	0.117571	1.129206
	5	0.010599	0.151735	0.020820	0.282078	0.038503	0.473877
	10	0.000485	0.006918	0.003814	0.053877	0.011768	0.160072
	15	2.211685e-05	0.000315	0.000722	0.010290	0.003851	0.054071
	20	1.008415e-06	1.438214e-05	0.000137	0.001965	0.001287	0.018264
	25	4.597817e-08	6.557470e-07	2.632413e-05	0.000375	0.000433	0.006169
	30	2.096352e-09	2.989846e-08	5.027658e-06	7.170423e-05	0.000146	0.002084
	35	9.558211e-11	1.363205e-09	9.602765e-07	1.369556e-05	4.937109e-05	0.000703
	40	4.358038e-12	6.215493e-11	1.834131e-07	2.615862e-06	1.667495e-05	0.000237
$\varepsilon = 0.14$	1	0.296761	4.381385	0.083690	0.915752	0.111610	1.022579
	5	0.068530	0.915983	0.016933	0.212009	0.034519	0.397209
	10	0.010003	0.129489	0.002635	0.034050	0.009714	0.121816
	15	0.001407	0.018305	0.000421	0.005468	0.002905	0.037359
	20	0.000199	0.002587	6.759178e-05	0.000878	0.000884	0.011457
	25	2.814736e-05	0.000365	1.085438e-05	0.000141	0.000270	0.003513
	30	3.979134e-06	5.171469e-05	1.743261e-06	2.265618e-05	8.294116e-05	0.001077
	35	5.625137e-07	7.310703e-06	2.799804e-07	3.638759e-06	2.543107e-05	0.000330
	40	7.952038e-08	1.033485e-06	4.496699e-08	5.844128e-07	7.798738e-06	0.000101

Table 1**.** The choice of parameters for the Jacobi, TSS and TS iteration methods.

6. Conclusions

In this paper, the steady state vector of regularized linear system of block circulant matrices is estimated using TS method. It is also proved that the coefficient matrix of regularized linear system (4) is positive definite. Theoretical analysis shows that the iterative solution of TS method converges to the unique solution of the system for a wide range of the parameter α . A bound for the spectral radius of the iteration matrix is derived and the numerical example involving the contraction factor α^* which minimizes the upper bound. From the numerical results, is is demonstrated that how well the iterative solution of present splitting method is superiored when compared with the existing methods.

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