JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 3566-3570
https://publishoa.com
ISSN: 1309-3452

Some Identities on Finite Sums of Product of Fibonacci and Lucas Numbers in Chebyshev Polynomials of Second Kind

Mr. Jugal Kishore[1], Dr. Vipin Verma[2]
1,2 Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara 144411, Punjab (INDIA)
(Email: jkish11111@gmail.com, vipin_soni2406@rediffmail.com)"

Received 2022 April 02**; Revised** 2022 May 20**; Accepted** 2022 June 18**.**

Abstract
This paper shall attempt to introduce some identities involving finite sums of the product of the Fibonacci, the Lucas numbers and the Complex Fibonacci numbers in the derivatives of Chebyshev polynomials of second kind using computational method.

Keywords: Chebyshev Polynomials, Lucas numbers, Fibonacci Numbers, Complex Fibonacci Numbers.

1. Introduction

The Fibonacci numbers (F_n) and Lucas Numbers (L_n) are defined by the second-order Linear recursive relations

$$F_n = F_{n-1} + F_{n-2}, n \geq 2 \quad \text{with} \quad F_0 = 0 \quad \text{and } F_1 = 1 \tag{1}$$

and

$$L_n = L_{n-1} + L_{n-2}, n \geq 2 \text{ with } L_0 = 2 \text{ and } L_1 = 1 \tag{2}$$

Again, the Complex Fibonacci numbers [7] are represented by the recursive relation as follows

$$F^*{}_n = F^*{}_{n-1} + F^*{}_{n-2}, n \geq 2 \quad \text{with} \quad F^*{}_0 = i \quad \text{and } F^*{}_1 = 1 + i \tag{3}$$

where $i^2 = -1$ and

$$F^*{}_n = F_n + i\, F_{n+1} \tag{4}$$

The Chebyshev polynomials of first, second, third, and fourth kind [2][5] defined recursively for integer $n \geq 1$, are as under:

$$\mathcal{T}_n(z) = 2z\,\mathcal{T}_{n-1}(z) - \mathcal{T}_{n-2}(z) \quad \text{with} \quad \mathcal{T}_0(z) = 1 \text{ and } \mathcal{T}_1(z) = z \tag{5}$$

$$\mathcal{U}_n(z) = 2z\,\mathcal{U}_{n-1}(z) - \mathcal{U}_{n-2}(z) \quad \text{with} \quad \mathcal{U}_0(z) = 1 \text{ and } \mathcal{U}_1(z) = 2z \tag{6}$$

$$\mathcal{V}_n(z) = 2z\,\mathcal{V}_{n-1}(z) - \mathcal{V}_{n-2}(z) \quad \text{with } \mathcal{V}_0(z) = 1 \text{ and } \mathcal{V}_1(z) = 2z - 1 \tag{7}$$

$$\mathcal{W}_n(z) = 2z\,\mathcal{W}_{n-1}(z) - \mathcal{W}_{n-2}(z) \text{ with } \mathcal{W}_0(z) = 1 \text{ and } \mathcal{W}_1(z) = 2z + 1 \tag{8}$$

These Second-order linear recurrence sequences in turn leads to following general formulae [1][5]

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 3566-3570
https://publishoa.com
ISSN: 1309-3452

$$\mathcal{T}_n(z) = \frac{1}{2}\left[\left(z + \sqrt{z^2 - 1}\right)^n + \left(z - \sqrt{z^2 - 1}\right)^n\right] \tag{9}$$

$$\mathcal{U}_n(z) = \frac{1}{2\sqrt{z^2 - 1}}\left[\left(z + \sqrt{z^2 - 1}\right)^{n+1} - \left(z - \sqrt{z^2 - 1}\right)^{n+1}\right] \tag{10}$$

Utilizing above discussed recurrence relations, it can be easily observed that, for integer $n \geq 0$, we have ([1]-[3], [5]),

$$\mathcal{V}_n(z) = \mathcal{U}_n(z) - \mathcal{U}_{n-1}(z) \tag{11}$$

$$\mathcal{W}_n(z) = \mathcal{U}_n(z) + \mathcal{U}_{n-1}(z) \tag{12}$$

$$\mathcal{W}_n(z) = (-1)^n \ \mathcal{V}_n(-z) \tag{13}$$

The Chebyshev polynomials properties were given by many authors for instance Zhang [2] has studied the finite sums of the product of Chebyshev polynomials, Fibonacci and Lucas Numbers and derived interesting results, particularly

$$\sum_{d_1+d_2+\cdots+d_{r+1}=n} \mathcal{U}_{d_1}(z) \cdot \mathcal{U}_{d_2}(z) \cdots \mathcal{U}_{d_{r+1}}(z) = \frac{1}{2^r \ r!} \mathcal{U}_{n+r}^r(z) \tag{14}$$

where$\mathcal{U}_n^r(z)$ denotes the r^{th} derivative of $\mathcal{U}_n(z)$ w.r.t x and the sum runs over all the r+1-dimensional non-negative integral coordinates $(d_1, d_2, \cdots, d_{r+1})$ such that $d_1 + d_2 + \cdots + d_{r+1} = n$

In the same line, this paper shall attempt to introduce some more identities, involving finite sums of the product of the Fibonacci and the Lucas numbers and the derivative of the second kind Chebyshev polynomials using computational method. We have following main results.

Theorem 1: For $n, r \geq 0$,

$$\sum_{d_1+d_2+\cdots+d_{r+1}=n} F_{2d_1+1} \cdot F_{2d_2+1} \cdots F_{2d_{r+1}+1} = \frac{1}{2^r \ r!} \sum_{j=0}^{n} (-1)^j \binom{r+1}{j} \mathcal{U}_{n-i+j}^r\left(\frac{3}{2}\right)$$

where $\binom{r+1}{j} = 0$ *for j > r+1*

Theorem 2: For $n, r \geq 0$,

$$\sum_{d_1+d_2+\cdots+d_{r+1}=n} L_{2d_1+1} \cdot L_{2d_2+1} \cdots L_{2d_{r+1}+1} = \frac{1}{2^r \ r!} \sum_{j=0}^{n} \binom{r+1}{j} \mathcal{U}_{n-i+j}^r\left(\frac{3}{2}\right)$$

where $\binom{r+1}{j} = 0$ *for j > r+1*

Theorem 3: For $n, r \geq 0$,

$$\sum_{d_1+d_2+\cdots+d_{r+1}=n} {F^*}_{2d_1+1} \cdot {F^*}_{2d_2+1} \cdots {F^*}_{2d_{r+1}+1} = \frac{(i^{n+1})^{r+1}}{2^r \ r!} \sum_{j=0}^{n} (-1)^j \binom{r+1}{j} \mathcal{U}_{n-i+j}^r\left(-\frac{i}{2}\right)$$

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 3566-3570
https://publishoa.com
ISSN: 1309-3452

$$= \frac{1}{(i^{n-1})^{r+1}\, 2^r\; r!}\sum_{j=0}^{n}(-1)^j \binom{r+1}{j} \mathcal{U}_{n-i+j}^{r}\left(\frac{i}{2}\right)$$

where $\binom{r+1}{j} = 0$ *for j > r+1* and $F^*{}_n$ *is a Complex Fibonacci number*.

Corollary 1: For $n, r \geq 0$,

$$\sum_{d_1+d_2+\cdots+d_{r+1}=n} F_{-(2d_1+1)}\cdot F_{-(2d_2+1)}\cdots F_{-(2d_{r+1}+1)} = \frac{1}{2^r\; r!}\sum_{j=0}^{n}(-1)^j \binom{r+1}{j} \mathcal{U}_{n-i+j}^{r}\left(\frac{3}{2}\right)$$

where $\binom{r+1}{j} = 0$ *for j > r+1*

Corollary 2: For $n, r \geq 0$,

$$\sum_{d_1+d_2+\cdots+d_{r+1}=n} L_{-(2d_1+1)}\cdot L_{-(2d_2+1)}\cdots L_{-(2d_{r+1}+1)} = \frac{(-1)^{r+1}}{2^r\; r!}\sum_{j=0}^{n} \binom{r+1}{j} \mathcal{U}_{n-i+j}^{r}\left(\frac{3}{2}\right)$$

where $\binom{r+1}{j} = 0$ *for j > r+1*

Corollary 3: For $n, r \geq 0$,

$$\sum_{d_1+d_2+\cdots+d_{r+1}=n} F^*{}_{-(2d_1+1)}\cdot F^*{}_{-(2d_2+1)}\cdots F^*{}_{-(2d_{r+1}+1)} = \frac{((-i)^{n+1})^{r+1}}{2^r\; r!}\sum_{j=0}^{n}(-1)^j \binom{r+1}{j} \mathcal{U}_{n-i+j}^{r}\left(\frac{i}{2}\right)$$

$$= \frac{1}{((-i)^{n-1})^{r+1}\, 2^r\; r!}\sum_{j=0}^{n}(-1)^j \binom{r+1}{j} \mathcal{U}_{n-i+j}^{r}\left(-\frac{i}{2}\right)$$

where $\binom{r+1}{j} = 0$ *for j > r+1* and $F^*{}_n$ *is a Complex Fibonacci number*.

2. Proof of the Theorems and Corollaries.

From [6], we have

$$\sum_{d_1+d_2+\cdots+d_{r+1}=n} \mathcal{V}_{d_1}(z)\cdot \mathcal{V}_{d_2}(z)\cdots \mathcal{V}_{d_{r+1}}(z) = \frac{1}{2^r\; r!}\sum_{j=0}^{n}(-1)^j \binom{r+1}{j} \mathcal{U}_{n-i+j}^{r}(z) \qquad (15)$$

$$\sum_{d_1+d_2+\cdots+d_{r+1}=n} \mathcal{W}_{d_1}(z)\cdot \mathcal{W}_{d_2}(z)\cdots \mathcal{W}_{d_{r+1}}(z) = \frac{1}{2^r\; r!}\sum_{jj=0}^{n} \binom{r+1}{j} \mathcal{U}_{n-i+j}^{r}(z) \qquad (16)$$

where all sums in (15)-(16) runs over all non-negative integers $(d_1, d_2, \cdots, d_{r+1})$ such that $d_1 + d_2 + \cdots + d_{r+1} = n$ with $\binom{r+1}{j} = 0$ *for j > r+1.*

For proof of theorem 1, take $z = \frac{3}{2}$ in equation (15), we have

$$\sum_{d_1+d_2+\cdots+d_{r+1}=n} \mathcal{V}_{d_1}\left(\frac{3}{2}\right) \cdot \mathcal{V}_{d_2}\left(\frac{3}{2}\right) \cdots \mathcal{V}_{d_{r+1}}\left(\frac{3}{2}\right) = \frac{1}{2^r\ r!} \sum_{j=0}^{n} (-1)^j \binom{r+1}{j} \mathcal{U}_{n-i+j}^{r}\left(\frac{3}{2}\right) \quad (17)$$

Using , $\mathcal{U}_n\left(\frac{3}{2}\right) = F_{2n+2}$ in equation (11)to get $\mathcal{W}_n\left(\frac{3}{2}\right) = F_{2n+1}$

So, we have from (17)

$$\sum_{d_1+d_2+\cdots+d_{r+1}=n} F_{2d_1+1} \cdot F_{2d_2+1} \cdots F_{2d_{r+1}+1} = \frac{1}{2^r\ r!} \sum_{j=0}^{n} (-1)^j \binom{r+1}{j} \mathcal{U}_{n-i+j}^{r}\left(\frac{3}{2}\right)$$

Hence the theorem 1 is proved. ■

For theorem 2, similarly, taking $z = \frac{3}{2}$ in equation (16), we have

$$\sum_{d_1+d_2+\cdots+d_{r+1}=n} \mathcal{W}_{d}\left(\frac{3}{2}\right) \cdot \mathcal{W}_{d_2}\left(\frac{3}{2}\right) \cdots \mathcal{W}_{d_{r+1}}\left(\frac{3}{2}\right) = \frac{1}{2^r\ r!} \sum_{j=0}^{n} \binom{r+1}{j} \mathcal{U}_{n-i+j}^{r}\left(\frac{3}{2}\right) \quad (18)$$

Using , $\mathcal{U}_n\left(\frac{3}{2}\right) = F_{2n+2}$ in equation (8) to get $\mathcal{W}_n\left(\frac{3}{2}\right) = L_{2n+1}$

we have

$$\sum_{d_1+d_2+\cdots+d_{r+1}=n} L_{2d_1+1} \cdot L_{2d_2+1} \cdots L_{2d_{r+1}+1} = \frac{1}{2^r\ r!} \sum_{j=0}^{n} \binom{r+1}{j} \mathcal{U}_{n-i+j}^{r}\left(\frac{3}{2}\right)$$

Hence the theorem 2 is proved. ■

For proof of the theorem 3, take $z = -\frac{i}{2}$ in equation (15), and $z = \frac{i}{2}$ in (16) , we have

$$\sum_{d_1+d_2+\cdots+d_{r+1}=n} \mathcal{V}_{d_1}\left(-\frac{i}{2}\right) \cdot \mathcal{V}_{d_2}\left(-\frac{i}{2}\right) \cdots \mathcal{V}_{d_{r+1}}\left(-\frac{i}{2}\right) = \frac{1}{2^r\ r!} \sum_{j=0}^{n} (-1)^j \binom{r+1}{j} \mathcal{U}_{n-i+j}^{r}\left(-\frac{i}{2}\right) \quad (19)$$

$$\sum_{d_1+d_2+\cdots+d_{r+1}=n} \mathcal{W}_{d_1}\left(\frac{i}{2}\right) \cdot \mathcal{W}_{d_2}\left(\frac{i}{2}\right) \cdots \mathcal{W}_{d_{r+1}}\left(\frac{i}{2}\right) = \frac{1}{2^r\ r!} \sum_{j=0}^{n} \binom{r+1}{j} \mathcal{U}_{n-i+j}^{r}\left(\frac{i}{2}\right) \quad (20)$$

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 3566-3570
https://publishoa.com
ISSN: 1309-3452

Using , $\mathcal{U}_n\left(\frac{i}{2}\right) = i^n F_{n+1}$ in equation (12) to get $\mathcal{W}_n\left(\frac{i}{2}\right) = i^{n-1} F_n^*$
and using this in turn in (13) we get, $\mathcal{V}_n\left(-\frac{i}{2}\right) = \frac{F_n^*}{i^{n+1}}$.

Therefore (15) and (16) reduces to

$$\sum_{d_1+d_2+\cdots+d_{r+1}=n} {F^*}_{2d_1+1}\cdot {F^*}_{2d_2+1}\cdots {F^*}_{2d_{r+1}+1} = \frac{(i^{n+1})^{r+1}}{2^r\ r!}\sum_{j=0}^{n}(-1)^j\binom{r+1}{j}\mathcal{U}_{n-i+j}^r\left(-\frac{i}{2}\right) \quad (21)$$

$$\sum_{d_1+d_2+\cdots+d_{r+1}=n} {F^*}_{2d_1+1}\cdot {F^*}_{2d_2+1}\cdots {F^*}_{2d_{r+1}+1} = \frac{1}{(i^{n-1})^{r+1}\ 2^r\ r!}\sum_{j=0}^{n}(-1)^j\binom{r+1}{j}\mathcal{U}_{n-i+j}^r\left(\frac{i}{2}\right) \quad (22)$$

This establishes the theorem 3. ■

For proof of Corollary 1 and 2, using $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^n L_n$ in theorem 1 and 2 respectively, we get the desired results. Again, the Corollary 3 can be established by taking the conjugate of ${F^*}_n$ in theorem 3 and using ${F^*}_{-n} = (-1)^{n+1}\overline{{F^*}_n}$, where $\overline{{F^*}_n}$ is conjugate of ${F^*}_n$. ■

References

1. Handscomb, D.C., Mason, J, C.: Chebyshev Polynomials. Chapman & Hall/CRC, Boca Raton (2002)
2. Zhang, W.: On Chebyshev polynomials and Fibonacci numbers. Fibonacci Q. 40(5) ,424-428,2002.
3. Zhang, W.: Some identities involving the Fibonacci numbers and Lucas numbers. Fibonacci Q. 42 ,149-154 (2004).
4. Siyi, W.: Some new identities of Chebyshev polynomials and their applications. *Adv Differ Equ*. 2015, 355(2015).
5. George, B., Doman, S: The Classical Orthogonal Polynomials, Chap. 05-08, pp. 61-109, World Scientific Publishing Co Pte Ltd (2016).
6. Kim, T., Dolgy, D.V., Kim, D.S. *et al:* representation by several orthogonal polynomials for sums of finite product of Chebyshev polynomials of the first, third, and fourth kind. *Adv Differ Equ*. 2019, 110(2019).
7. B. Prasad, Dual complex Fibonacci p-numbers, Chaos, Solitons & Fractals, Volume 145, (2021).