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Equitable Total Coloring of Star Graph Families

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ABSTRACT

A proper total coloring of a graph G is said to be equitable, if the numberof elements (vertices and edges) in any two color classes differ by at most one, for which therequired minimum number of colors is called the equitable total chromatic number. In thispaper, we determine equitable total chromatic numbers of double star graph, bistar graph and splitting graph of star graph.

Key Words: Star graph, bistar graph, double star graph, splitting graph, equitable total coloring

1. INTRODUCTION

In this paper, we consider only finite undirected graphs without loops or multiple edges.Let G = (V (G), E(G)) be a graph with vertex set V (G) and edge set E(G). In 1994, the concept of total coloring χ (G) was introducedby Behzad [1] and Vizing [9]. A total coloring of a graph G is anassignment of colors to both the vertices and edges of G, such that no twoadjacent or incident vertices and edges of G are received the same colors.They both conjectured that for any graph G the following inequality holds: $\Delta(G)+1\leq\chi^{''}(G)\leq\Delta(G)+2$, where Δ(G) is the maximum degree of G.It is clear that $\Delta(G)+1$ is the possible lower bound.In 1994, Fu [3] first introduced the concept of equitable totalcoloring and the equitable total chromatic number of a graph. Gong Kun et.al [2] provedsome results on the equitable total chromatic number of $W_n\vee K_n, F_n\vee K_n$ and $S_m\vee K_n$. In2012, Ma Gang and ma Ming [7] proved some results concerning the equitable total chromaticnumber of $P_m\vee S_n, P_m\vee F_n$ and $P_m\vee W_n$. Ma Gang , zhangzhong fu [8] proved that on the equitable total coloringof multiple join graph. Jayaraman and muthuramakrishnan [4] proved that equitable total chromatic number of splitting graphs. Jayaraman and muthuramakrishnan [5] proved that equitable total chromatic number middle graph of path, middle graph of cycle, total graph of path and total graph of cycle. Veninstinevivik et.al [10] proved an algorithmic approach to equitable totalchromatic number of wheel graph, Gear graph, Helm graph and sunlet graph.Wang Wei-fan [11] proved thatequitable total coloring of graphs with maximum degree 3. Zhang Zhong-fu [12] proved that on the equitable total coloring ofsome join graphs.

2. PREMINARIES

Definition 2.1. A tree containing exactly one vertex that is not a pendent vertex is called a star graph $K_{1,n}$.

Definition 2.2. The *double star graph* $K_{1,n,n}$ [6] is a tree obtained from the star $K_{1,n}$ by adding a new pendant edge of the exiting n pendant vertices. It has $2n+1$ vertices and $2n$ edges.

Definition 2.3. The *bistar* $B_{n,n}$ is graph obtained by joining the root vertices of two copies of $K_{1,n}$ by an edge.

Definition 2.4. The *splitting graph* $S^{'}(G)$ [3] of a graph G is obtained from adding to each vertex v, a new vertex $v^{'}$ such that $v^{'}$ is adjacent to every vertex that is adjacent to v in G that is $N(v) = N(v^{'})$.

Definition 2.5. An *equitable total coloring*of a graph G is the coloring of all its vertices and edges in which the number of elements in any two color classes differ by atmost one. The minimum number of colors required is called its *equitable total chromatic number*and it is denoted by $\chi_{et}(G)$.

In other words, For a simple graph G ,let f be a proper *k*-total coloring of G,

$\|T_i| - |T_j\| \le 1$, for $i, j = 1,2,3,....,k$.

The partition $T_i = \{V_i \cup E_i : 1 \le i \le k\}$ is called a *k*-equitable total coloring and

$\chi_{et}(G) = \min\{k / k - \text{equitable total coloring of} G\}$

is called the *equitable total chromatic number*of G, where $\forall\, x \in T_i = V_i \cup E_i$, $f(x) = i,\ i = 1,2,.....,k$.

Conjecture 2.6. For any simple graph G, $\chi_{et}(G) \le \Delta(G) + 2$.

Conjecture 2.7. For any simple graph G, $\chi_{et}(G) \ge \chi^{''}(G) \ge \Delta(G) + 1$.

3. MAIN RESULTS

Theorem 3.1. Let $K_{1,n,n}$ be the double star graph. Then itsequitable total chromatic number is $\chi_{et}\left(K_{1,n,n}\right) = n+1, n \ge 2.$

Proof:

Let

$V(K_{1,n,n}) = \{u\} \bigcup \{u_i : 1 \le i \le n\} \bigcup \{v_i : 1 \le i \le n\}$ and $E(K_{1,n,n}) = \{e_i : 1 \le i \le n\} \bigcup \{f_i : 1 \le i \le n\}$, where $\{e_i : 1 \le i \le n\}$ be the edges $\{uu_i : 1 \le i \le n\}$ and $\{f_i : 1 \le i \le n\}$ be the edges $\{u_i v_i : 1 \le i \le n\}$. Now we partition the edge and vertex sets in $K_{1,n,n}$ as follows.

$T_1 = \{e_1, u_n, f_{n-1}, v_{n-2}\}$,

$T_2 = \{e_2, u_1, f_n, v_{n-1}\}$,

$T_3 = \{e_3, u_2, v_n\}$,

$T_4 = \{e_4, u_3, f_1\}$,

....

....

....

$T_{n-1} = \{e_{n-1}, u_{n-2}, f_{n-4}, v_{n-5}\}$,

$T_n = \{e_n, u_{n-1}, f_{n-3} v_{n-4}\}$,

$T_{n+1} = \{u, f_{n-2}, v_{n-3}\}$,

Clearly $T_1, T_2, T_3, ..., T_n$ and T_{n+1} are independent sets of $K_{1,n,n}$. Also $|T_1| = |T_2| = |T_i| = 4$ $(5 \le i \le n)$ and $|T_3| = |T_4| = |T_{n+1}| = 3,$ Its satisfies the inequality $\|T_i\| - \|T_j\| \le 1,$ for $i \ne j$ Therefore the graph $K_{1,n,n}$ is equitably total colored with $n+1$ colors. This implies that $\chi_{et}(K_{1,n,n}) \le n+1$. Further, since $\Delta = n$ .We have $\chi_{et}(K_{1,n,n}) \ge \Delta(K_{1,n,n}) + 1 \ge n+1$. Hence $\chi_{et}(K_{1,n,n}) = n+1$.

Example1: Consider the double star graph with 5 vertices. Let the graph $K_{1,5,5}$ is equitable total colored the above method of coloring pattern as given in theorem 3.1, the colors 1, 2, 3, 4, 5, 6 are assigned to splitting graph of star $K_{1,5,5}$ is shown in figure 1. The vertices and edges are partition into the color classes $T_1, T_2, T_3, T_4, T_5, T_6$ and it is independent sets of $K_{1,5,5}$. Also $|T_1| = |T_2| = |T_5| = |T_6| = ... = |T_n| = 4$ and $|T_3| = |T_4| = ... = |T_{n+1}| = 3$. Its satisfies the inequality $\|T_i\| - \|T_j\| \le 1,$ for every pair (i, j). Therefore the graph $K_{1,5,5}$ is equitably total colored with 6 colors. This implies that $\chi_{et}(K_{1,5,5}) \le 6$. Further, since $\Delta = 5$,we have $\chi_{et}(K_{1,5,5}) \ge 5+1 \ge 6$. Hence $\chi_{et}(K_{1,5,5}) = 6$.

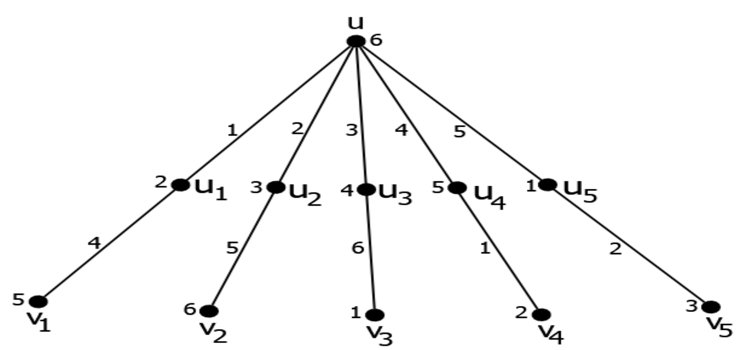

Figure 1: Equitable total coloring of double star graph

Theorem 3.2. Let $S'(K_{1,n})$ be the splitting graph of star graph. Then itsequitable total chromatic number is $\chi_{et}(S'(K_{1,n})) = 2n+1, n \ge 2.$

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Proof:

Let $V(K_{1,n})=\{v\}\cup\{v_i:1\le i\le n\}$, where $\{v_i:1\le i\le n\}$ be the pendent vertices and $\{v\}$ be the root vertex of $K_{1,n}$ and $E(K_{1,n})=\{e_i:1\le i\le n\}$, where $\{e_i:1\le i\le n\}$ be the edges $\{vv_i:1\le i\le n\}$. Now construct the splitting graph of star graph, introduce the new vertices $\{v^{'}\}$ and $\{v_i^{'}:1\le i\le n\}$ corresponding to the vertices $\{v\}$ and $\{v_i:1\le i\le n\}$ of $K_{1,n}$, which are added to obtained splitting graph of star $S^{'}(K_{1,n})$.

In $S^{'}(K_{1,n})$, the vertex set is given by

$$V\left(S^{'}(K_{1,n})\right)=\{v\}\cup\{v^{'}\}\cup\{v_i:1\le i\le n\}\cup\{v_i^{'}:1\le i\le n\},$$

The edge set be represented by

$$E\left(S^{'}(K_{1,n})\right)=\{e_i:1\le i\le n\}\cup\{e_i^{'}:1\le i\le n\}\cup\{e_i^{''}:1\le i\le n\},$$

where $\{e_i:1\le i\le n\}$ is an edges $\{vv_i:1\le i\le n\}$, $\{e_i^{'}:1\le i\le n\}$ is an edges $\{vv_i^{'}:1\le i\le n\}$, and $\{e_i^{''}:1\le i\le n\}$ is an edges $\{v^{'}v_i:1\le i\le n\}$. Now we partition the vertex set and edge set of splitting graph of path $S^{'}(K_{1,n})$ as follows.

$T_1=\{v,v^{'}\}$,

$T_2=\{e_1^{'},e_1^{''},v_n\}$,

$T_3=\{e_2^{'},e_2^{''},v_1^{'}\}$,

$T_4=\{e_3^{'},e_3^{''},v_2^{'}\}$,

$T_5=\{e_4^{'},e_4^{''},v_3^{'}\}$,

....

....

$T_{n-1}=\{e_{n-2}^{'},e_{n-2}^{''},v_{n-3}^{'}\}$,

$T_n=\{e_{n-1}^{'},e_{n-1}^{''},v_{n-2}^{'}\}$,

$T_{n+1}=\{e_n^{'},e_n^{''},v_{n-1}^{'}\}$,

$T_{n+2}=\{e_1,v_n^{'}\}$,

$T_{n+3}=\{e_2,v_1\}$,

$T_{n+4}=\{e_3,v_2\}$,

....

....

$T_{2n-1}=\{e_{n-2},v_{n-3}\}$,

$T_{2n}=\{e_{n-1},v_{n-2}\}$,

$T_{2n+1}=\{e_n,v_{n-1}\}$,

Clearly $T_1,T_2,T_3,T_4,...,T_{2n+1}$ are independent sets of $S^{'}(K_{1,n})$. Also $|T_2|=|T_3|=|T_4|=....=|T_{n+1}|=3$ and $|T_1|=|T_{n+2}|=|T_{n+3}|=....=|T_{2n}|=|T_{2n+1}|=2$. Its satisfies the inequality $\|T_i\|-\|T_j\|\le 1$, for $i\ne j$. Therefore the graph $S^{'}(K_{1,n})$ is equitably total colored with $2n+1$ colors. This implies that $\chi_{et}\left(S^{'}(K_{1,n})\right)\le 2n+1$. Further, since $\Delta=2n$. We have $\chi_{et}\left(S^{'}(K_{1,n})\right)=\chi^{''}\left(S^{'}(K_{1,n})\right)\ge \Delta(G)+1\ge 2n+1$. Hence $\chi_{et}\left(S^{'}(K_{1,n})\right)=2n+1$.

Example 2: *Consider the splitting graph of star graph* $S^{'}(K_{1,6})$. Let the graph $S^{'}(K_{1,6})$

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has equitable total colored the above method of coloring pattern as given in theorem 3.2, the colors 1, 2, 3, 4,…,13 are assigned to splitting graph of star $S^{'}(K_{1,6})$ is shown in figure 2. The vertices and edges are partition into the color classes $T_1, T_2, T_3, T_4, ... T_{13}$ and it is independent sets of $S^{'}(K_{1,6})$. Also $|T_2| = |T_3| = |T_4| = |T_5| = |T_6| = |T_7| = 3$ and $|T_1| = |T_8| = |T_9| = |T_{10}| = |T_{11}| = |T_{12}| = |T_{13}| = 2$. Its satisfies the inequality $\| T_i \| - \| T_j \| \le 1$, for every pair (i, j). Therefore the graph $S^{'}(K_{1,6})$ is equitably total colored with 13 colors. This implies that $\chi_{et}\left(S^{'}(K_{1,6})\right) \le 13$. Further, since $\Delta = 12$, we have $\chi_{et}\left(S^{'}(K_{1,6})\right) \ge 12 + 1 \ge 13$. Hence $\chi_{et}\left(S^{'}(K_{1,6})\right) = 13$.

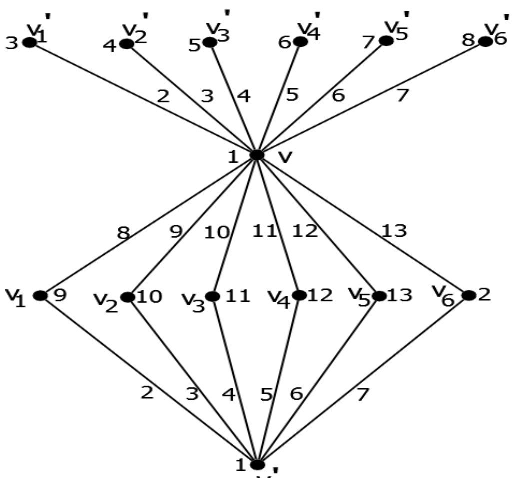

Figure 2: Equitable total coloring of splitting graph of cycle $S^{'}(K_{1,6})$

Theorem 3.3. Let $B_{n,n}$ be the bistar graph. Then itsequitable total chromatic number is $\chi_{et}\left(B_{n,n}\right) = \Delta(B_{n,n}) + 1.$

P**roof:** In $B_{n,n}$,the vertex set is given by $V\left(B_{n,n}\right) = \{u\} \cup \{v\} \cup \{u_i : 1 \le i \le n\} \cup \{v_i : 1 \le i \le n\}$,
The edge set be represented by
$E\left(B_{n,n}\right) = \{e\} \cup \{e_i : 1 \le i \le n\} \cup \{e_i^{'} : 1 \le i \le n\}$
,
where $\{e\}$ is an edge $\{uv\}$, $\{e_i : 1 \le i \le n\}$ is an edges $\{uu_i : 1 \le i \le n\}$, $\{e_i^{'} : 1 \le i \le n\}$ is an edges $\{vv_i : 1 \le i \le n\}$. Now we partition the vertex set and edge set of splitting graph of path $B_{n,n}$ as follows.

$T_1 = \{u, v_n, e_1^{'}\}$,

$T_2 = \{v_1, e_1, e_2^{'}\}$,

$T_3 = \{u_1, v_2, e_2, e_3^{'}\}$,

$T_4 = \{u_2, v_3, e_3, e_4^{'}\}$,

$T_5 = \{u_3, v_4, e_4, e_5^{'}\}$,

....

....

$T_{\Delta-1} = \{u_{n-2}, e_{n-1}, e_n^{'}\}$,

$T_{\Delta} = \{v, u_{n-1}, e_n\}$,

$T_{\Delta+1} = \{u_n, v_{n-1}, e\}$,

Clearly $T_1, T_2, T_3, T_4, ..., T_{2n+1}$ are independent sets of $B_{n,n}$. Also $|T_1| = |T_2| = ... = |T_{\Delta-1}| = |T_{\Delta}| = |T_{\Delta+1}| = 3$ and $|T_3| = |T_4| = |T_5| = 4$. Its satisfies the inequality $\|T_i\| - \|T_j\| \le 1$, for $i \ne j$. Therefore the graph $B_{n,n}$ is equitably total colored with $\Delta(B_{n,n})+1$ colors. This implies that $\chi_{et}\left(B_{n,n}\right) \le \Delta(B_{n,n})+1$. Further,We have $\chi_{et}\left(B_{n,n}\right) \ge \Delta(B_{n,n})+1$. Hence $\chi_{et}\left(B_{n,n}\right) = \Delta(B_{n,n})+1.$

Example 3: Consider the bistar graph with 5 vertices

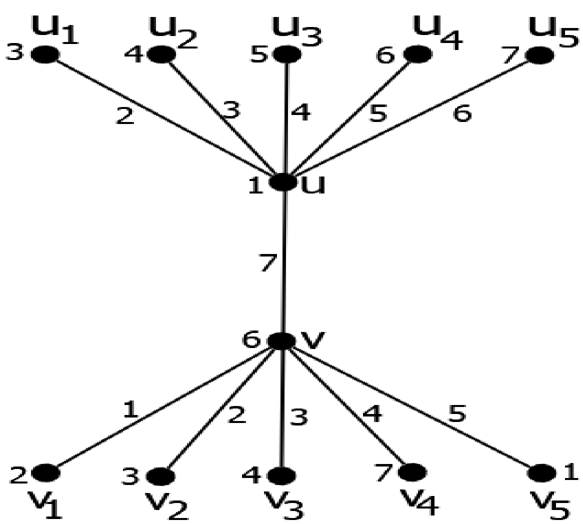

Figure 3: Equitable total coloring of bistar graph

Let the graph $B_{5,5}$ is equitable total colored the above method of coloring pattern as given in theorem 3.3, the colors 1, 2, 3, 4, 5, 6, 7are assigned to splitting graph of star $B_{5,5}$ is shown in figure 1. The vertices and edges are partition into the color classes $T_1, T_2, T_3, T_4, T_5, T_6$ and it is independent sets of $K_{1,5,5}$. Also $|T_1| = |T_2| = |T_5| = |T_6| = ... = |T_n| = 4$ and $|T_3| = |T_4| = ... = |T_{n+1}| = 3$. Its satisfies the inequality $\|T_i\| - \|T_j\| \le 1$, for every pair (i, j). Thus the graph $B_{5,5}$ is equitably total colored with 6 colors. This implies that $\chi_{et}\left(B_{5,5}\right) \le 7$. Further, since $\Delta = 6$,we have $\chi_{et}\left(B_{5,5}\right) \ge 6+1 \ge 7$.Hence $\chi_{et}\left(B_{5,5}\right) = 7.$

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