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Isometric Path Partition Number of Corona Product of Graphs

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Abstract

The collection of isometric paths that partition the vertex set of a graph G is an isometric path partition of G. The minimum cardinality of an isometric path partitionis called the isometric path partition number of G. In this paper, we computed an upper bound for the isometric path partition number of corona product of $G \odot H$ and investigate the isometric path partition number of corona product of G with path, cycle,complete graph and ladder graph.

AMS Subject Classification: 05C12; 05C38; 05C70

Keywords and Phrases: Isometric path; Path partition; Corona product

1 Introduction

Harary and Frucht [5] introduced a new product of two graphs G and H called corona productdenoted by $G \odot H$. Let $G = (V, E)$ and $H = (V_0, E_0)$ be the two graphs. The *corona product* of G and H is the graph $G \odot H$ is obtained by taking one copy of G = (V, E) called the centre graph and $|V(G)|$ copies of H, called the outer graph and by joining each vertex of the i^{th} copy of H to the i^{th} vertex of G, where $1 \leq i \leq |V(G)|$. In general, the corona product$G \odot H$ are neither commutative nor associative. For more properties on the corona product refer [1], [8], [9]. A *block* of G is a maximal subgraph without a cut-vertex. Throughout this paper, we consider an undirected connected graph without loops and multiple edges. We refer to Bondy and Murty [3] for the basic definitions and terminology.

We call a shortest path joining two vertices in a graph G as an *isometric* path. An *isometric subgraph* [7] of a graph G is defined as a subgraph H of G such that $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. A set of subgraphs $H_1, \ldots, H_k$ of a graph G is an *isometric cover* of G if each H_i, $1 \leq i \leq k$, is isometric in G and $\cup_{i=1}^{k} V(H_i) = V(G)$. An isometric cover of G is called as an *isometric partition* of G if $V(H_i) \cap V(H_j) = \varphi$ for $i \neq j$. An *isometric path partition* of G is defined as a set of isometric paths that partition the vertex set V of G. The *isometric path partition number*, denoted by $ip_p(G)$ is the cardinality of a minimum isometric path partition of G. The isometric path partition problem is to find a minimum isometric path partition of G. When the length of an isometric path is equal to the diameterof the graph, we denote such path as a *diametral isometric path* of a graph G.

Aggarwal et al. presented a study on the isometric path problem in [2]. Paul Manuel [6] proved NP-completeness of the isometric path partition problem and Fisher et al. [4] gave the lower bound for the same. In [6], Paul Manuel presented the isometric path partition number of multi-dimensional grid, torus and Benes network. In this paper, we compute the isometric path partition number of corona product of G with path, cycle, complete graph and ladder graph and an upper bound for the isometric path partition number of corona productof G and H.

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Proposition 1.1. *[4] If diam(G) denotes the diameter of a graph G,* $ip_p(G) \geq ip_c(G) \geq \left\lceil \frac{|V(G)|}{diam(G)+1} \right\rceil$

Proposition 1.2. *Let G be a connected graph with k cut-vertices. Let* $H_1, H_2, ..., H_k$ *be blocks of the graph G forming an isometric partition of G. If* P_i *is an isometric path partition of* H_i, $i \in [k]$ *such that every* $P_i^j \in \mathrm{P}_i$ *is a diametral isometric path and each cut-vertex in G is an internal vertex of some* P_i^j *, then* $ip_p(G) = \sum_{i=1}^{k} ip_p(H_i)$.

Proof. Since $H_1, H_2, ..., H_k$ is an isometric partition of G, $ip_p(G) \leq \sum_{i=1}^{k} ip_p(H_i)$. Suppose $ip_p(G) < \sum_{i=1}^{k} ip_p(H_i)$, then there exist atleast two diametral isometric paths $P_a^c \in \mathrm{P}_a$ and $P_b^d \in \mathrm{P}_b$ of H_a and H_b respectively, which may be combine to form an isometric path of G. Since each H_i is a block, the two cut-vertices in H_a and H_b must be the end vertices of the two paths P_a^c and P_b^d respectively, which is a contradiction to our assumption that each cut-vertex is an internal vertex. Hence the proof.

Corollary 1.1. *Let G be a connected graph with k cut-vertices. Let* $H_1, H_2, ..., H_k$ *be blocks of the graph G forming an isometric partition of G. If* P_i *is an isometric path partition of* H_i, $i \in [k]$ *such that atleast two adjacent cut-vertices of G are end vertices of some* P_i^j *, then* $ip_p(G) < \sum_{i=1}^{k} ip_p(H_i)$.

Observation 1. *Let G and H be the two graphs of order n and m respectively. Then the corona graph* $G \odot H$ *of order* $n(m+1)$ *contains exactly n cut-vertices and also the graph induced by all n cut-vertices is the graph G.*

One can easily verify that the union of the isometric path partitions of G and the n copies of H forms an isometric path partition of $G \odot H$. Hence the following result.

Proposition 1.3. *Let G and H be the two graphs of order n and m respectively. Then* $ip_p(G \odot H) \leq n(ip_p(H)) + ip_p(G)$.

2 Isometric path partition of $G \odot C_m$ and $G \odot P_m$

We start this section with the computation of isometric path partition of corona product of K_1 and a cycle graph or a path graph.

Proposition 2.1. *Let G be either a cycle graph or a path graph of order* $n \geq 6$. *Then* $ip_p(K_1 \odot G) = \left\lceil \frac{n+1}{3} \right\rceil$.

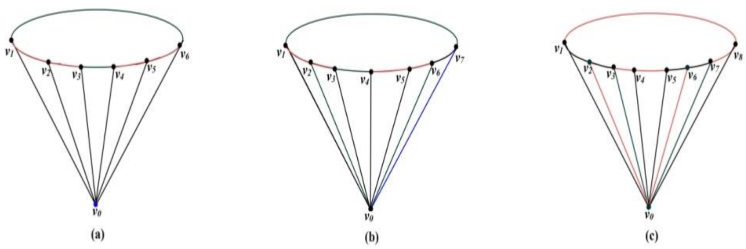

Figure 1: Isometric path partition of $K_1 \odot C_m$

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Proof. Observe that the diameter of $K_1 \odot G$ is 2. By Proposition 1.1, $ip_p(K_1 \odot G) \geq \left\lceil \frac{n+1}{3} \right\rceil$. Suppose G is a cycle C_n. Label the vertices of the cycle by $v_1, v_2, ..., v_n$ and the K_1-vertex by v_0. The following gives an isometric path partition of the required cardinality.

***Case* 1: $n \equiv 0$ (*mod* 3).**

$P_n = \{(v_1 - v_2 - v_3), (v_4 - v_5 - v_6), ..., (v_{n-2} - v_{n-1} - v_n), v_0\}$ (Refer figure 1(a)).

***Case* 2: $n \equiv 1$ (*mod* 3).**

$P_n = \{(v_1 - v_2 - v_3), (v_4 - v_5 - v_6), ..., (v_{n-3} - v_{n-2} - v_{n-1}), (v_n - v_0)\}$ (Refer figure 1(b)).

***Case* 3: $n \equiv 2$ (*mod* 3).**

$P_n = \{(v_3 - v_4 - v_5), (v_6 - v_7 - v_8), ..., (v_{n-5} - v_{n-4} - v_{n-3}), (v_2 - v_0 - v_{n-2}), (v_1 - v_n - v_{n-1})\}$(Refer figure 1(c)).

Suppose if G is a path P_n. Then for the cases $n \equiv 0, 1$ (*mod* 3), the above partition attains the lower bound. For $n \equiv 2$ (*mod* 3), $P_n = \{(v_2 - v_3 - v_4), (v_5 - v_6 - v_7), ..., (v_{n-3} - v_{n-2} - v_{n-1}), (v_n - v_0 - v_1)\}$ is the required partition. Hence the proof.

Proposition 2.2. *Let G be a connected graph of order n. Then*

$$ip_p(G \odot C_m) = \begin{cases} n + \left\lceil \frac{n}{2} \right\rceil, & m = 3 \\ n\left\lceil \frac{m}{3} \right\rceil + ip_p(G), & m > 3 \text{ and } m \equiv 0 \pmod 3 \\ n\left\lfloor \frac{m}{3} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil, & m \equiv 1 \pmod 3 \\ 2n + ip_p(G), & m = 5 \\ n\left\lceil \frac{m}{3} \right\rceil, & m > 5 \text{ and } m \equiv 2 \pmod 3 \end{cases}$$

Proof. Let $H_1, H_2, ..., H_n$ be an isometric partition of $G \odot C_m$ which are n copies of $K_1 \odot C_m$. Let P_i be the isometric path partition of H_i, $1 \leq i \leq n$. Then by Proposition 2.1, $ip_p(K_1 \odot C_m) = \left\lceil \frac{m+1}{3} \right\rceil$ for $m \geq 6$. Observe that each H_i is an isometric subgraph of diameter 2, hence by Proposition 1.5, it requires atmost $n\left\lceil \frac{m}{3} \right\rceil + ip_p(G)$ number of isometric paths to cover $G \odot C_m$. Now, we compute the isometric path partition number of $G \odot C_m$ in the following cases.

***Case* 1: $m \equiv 0$ (*mod* 3).**

***Subcase* 1.1: $m = 3$.**

Clearly $ip_p(K_1 \odot C_3) = 2$. Observe that each H_i is a K_4 that can be partitioned with P_2- paths (Refer figure 2(a)). A P_2-path through the cut-vertex of a H_i can be combined with a P_2-path of an adjacent H_j. Hence a

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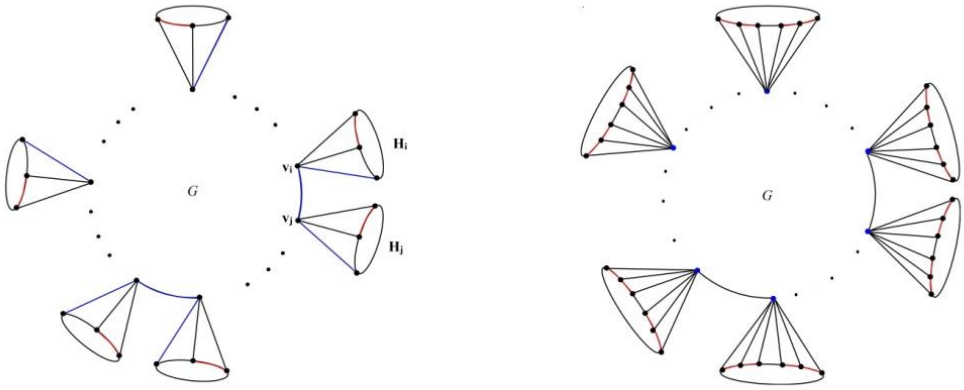
minimum of $n+\left\lceil \frac{n}{2} \right\rceil$ isometric paths are required to cover the $V(G \odot C_3)$ (Refer figure 2(a)).

***Subcase* 1.2: $m > 3$**

Observe that the elements of each P_i, $1 \leq i \leq n$ are $\left\lceil \frac{m}{3} \right\rceil$ diametral isometric paths and an isolated vertex (Refer *Case* 1 of Proposition 2.1). Therefore $ip_p(G \odot C_m) = n\left\lceil \frac{m}{3} \right\rceil + ip_p(G)$ (Refer figure 2(b)).



(a) (b)

Figure 2: Isometric path partition of $G \odot C_3$ and $G \odot C_6$

***Case* 2: $m \equiv 1\ (mod\ 3)$**

In this case, the elements of each P_i, $1 \leq i \leq n$ are $\left\lfloor \frac{m}{3} \right\rfloor$ diametral isometric paths and a P_2-path (Refer *Case* 2 of Proposition 2.1). Following the same lines of argument of *Subcase* 1.1, we obtain $ip_p(G \odot C_m) = n\left\lfloor \frac{m}{3} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil$ (Refer figure 3).

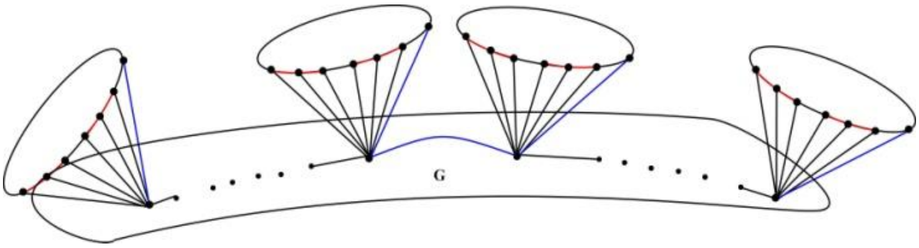


Figure 3: Isometric path partition of $G \odot C_7$

***Case* 3: $m \equiv 2\ (mod\ 3)$.**

***Subcase* 3.1: $m = 5$**

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Clearly $ip_p(K_1 \odot C_5) = 3$, where the isometric path partition includes a P_3-path, a P_2-path and an isolated vertex. In $G \odot C_5$, none of the P_2-paths belonging to H_i passes through the cut-vertex. Hence $ip_p(G \odot C_m) = 2n + ip_p(G)$ (Refer figure 4(a)).

***Subcase* 3.2: *m* > 5**

In this case, the elements of each P_i, $1 \le i \le n$ are $\left\lceil \frac{m}{3} \right\rceil$ diametral isometric paths. It is clear that each cut-vertex of $G \odot C_m$ is an internal vertex of some diametral isometric path. Hence, by Proposition 1.2, $ip_p(G) = \sum_{i=1}^{k} ip_p(H_i) = n\left\lceil \frac{m}{3} \right\rceil$ (Refer figure 4(b)).

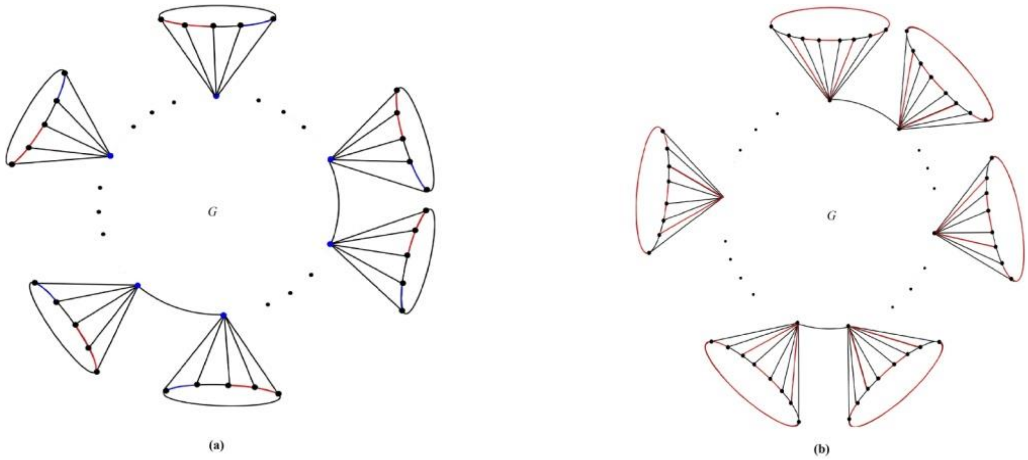


Figure 4: Isometric path partition of $G \odot C_5$ and $G \odot C_8$

By a similar argument for $G \odot P_m$, we obtain the following result.

Proposition 2.3. *Let G be a connected graph of order n. Then*

$$ip_p(G \odot P_m) = \begin{cases} n\left\lceil \frac{m}{3} \right\rceil + ip_p(G), & m \equiv 0 \pmod 3 \\ n\left\lfloor \frac{m}{3} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil, & m \equiv 1 \pmod 3 \\ n\left\lceil \frac{m}{3} \right\rceil, & m \equiv 2 \pmod 3 \end{cases}$$

3 Isometric path partition of $G \odot K_m$

In this section, we study the isometric path partition of the corona graph $G \odot K_m$. Observe that $K_1 \odot K_m$ is K_{m+1}. Clearly $ip_p(K_m) = \left\lceil \frac{m}{2} \right\rceil$ and the elements of the isometric path partition of K_m includes diametral isometric path (P_2-path) and an isolated vertex.

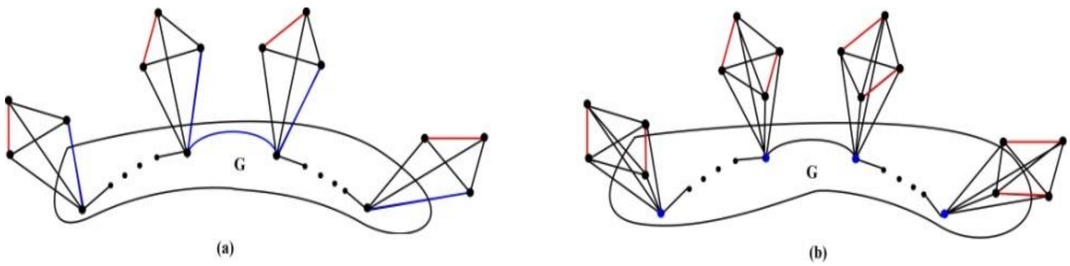
Proposition 3.1. *Let G be a connected graph of order n. Then*

$$ip_p(G \odot K_m) = \begin{cases} n\left\lceil \frac{m}{2} \right\rceil + ip_p(G), & m \text{ is even} \\ n\left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil, & m \text{ is odd} \end{cases}$$

Proof. Let $H_1, H_2, \ldots, H_n$ be an isometric partition of $G \odot K_m$, where each H_i is a complete graph K_m and P_i, the isometric path partition of H_i. When m is even, P_i includes $\left\lceil \frac{m}{2} \right\rceil$ P_2-paths and an isolated vertex, then it is clear that $ip_p(G \odot K_m) = n\left\lceil \frac{m}{2} \right\rceil + ip_p(G)$ (Refer figure 5(b)). Otherwise, P_i includes $\left\lfloor \frac{m}{2} \right\rfloor$ P_2-paths and observe that the cut-vertices of $G \odot K_m$ are end vertices of some P_2-paths. Therefore following the same lines of arguments of *Subcase* 1.1 of Proposition 2.2, we get $ip_p(G \odot K_m) = n\left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil$ (Refer figure 5(a)). Hence the proof.

Figure 5: Isometric path partition of $G \odot K_3$ and $G \odot K_4$

4 Isometric path partition of $G \odot L_m$



We begin with the computation of isometric path partition of corona product of K_1 and aladder graph.

Proposition 4.1. *Let L_n be the ladder graph of order $2n$. Then $ip_p(K_1 \odot L_n) = \left\lceil \frac{2n+1}{3} \right\rceil$.*

Proof. Diameter of $K_1 \odot L_n$ is 2 and hence $ip_p(K_1 \odot L_n) \geq \left\lceil \frac{2n+1}{3} \right\rceil$ by Proposition 1.1. Let us now label the vertices of the ladder graph by $x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{2n}$ and the K_1-vertexby x_0. We now construct an isometric path partition of the required cardinality to complete the proof.

***Case* 1: $n \equiv 0$ (*mod* 3).**

$\mathrm{P}_n = \{(x_1 - x_2 - x_3), (x_4 - x_5 - x_6), \ldots, (x_{2n-2} - x_{2n-1} - x_{2n}), x_0\}$ (Refer figure 6(a)).

***Case* 2: $n \equiv 1$ (*mod* 3).**

$\mathrm{P}_n = \{(x_2 - x_3 - x_4), (x_5 - x_6 - x_7), \ldots, (x_{n-2} - x_{n-1} - x_n), (x_1 - x_0 - x_{2n}), (x_{n+1} - x_{n+2} - x_{n+3}), (x_{n+4} - x_{n+5} - x_{n+6}), \ldots, (x_{2n-3} - x_{2n-2} - x_{2n-1})\}$ (Refer figure 6(b)).

***Case* 3: $n \equiv 2$ (*mod* 3).**

$\mathrm{P}_n = \{(x_{2n} - x_n - x_{n-1}), (x_{2n-3} - x_{n-3} - x_{n-4}), \ldots, (x_{2n-(n-2)} - x_2 - x_1), (x_{2n-1} - x_{2n-2} - x_{n-2}), (x_{2n-4} - x_{2n-5} - x_{n-5}), \ldots, (x_{2n-(n-4)} - x_{2n-(n-3)} - x_3), (x_{n+1} - x_0)\}$ (Refer figure 6(c)).

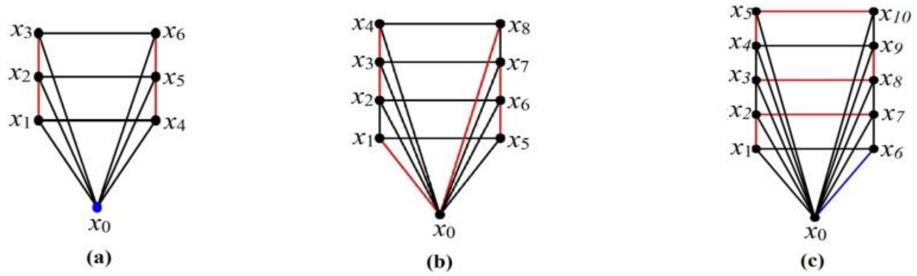


Figure 6: Isometric path partition of $K_1 \odot L_m$

Proposition 4.2. *Let G be a connected graph of order n. Then*

$$ip_p(G \odot L_m) = \begin{cases} n\left\lceil \frac{2m}{3} \right\rceil + ip_p(G), & m \equiv 0 \pmod 3 \\ n\left\lceil \frac{2m}{3} \right\rceil, & m \equiv 1 \pmod 3 \\ n\left\lfloor \frac{2m}{3} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil & m \equiv 2 \pmod 3 \end{cases}$$

Proof. The proof follows from Propositions 2.2 and 4.1 (Refer figure 7).

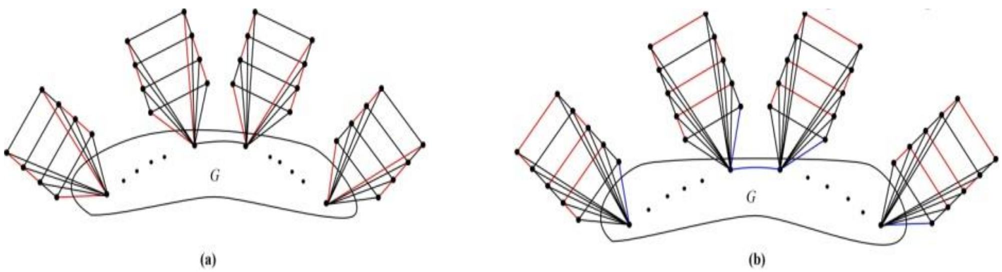


Figure 7: Isometric path partition of $K_1 \odot L_m$

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