Prime Labeling of $m$- Shadow Graph of Some Star Related Graphs

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**Received** 2022 March 25; **Revised** 2022 April 28; **Accepted** 2022 May 15.

**Abstract.** A graph $G = (V(G), E(G))$ is observed to admit prime labeling when the vertices of the graph are labeled with unique integral values from $[1, |V|]$ in a way that for every edge $uv$ the labels designated to $u$ and $v$ share no common positive factors except 1. In this research manuscript we examine and manifest that the $m$-shadow graph of some star related graphs such as $K_{1,n}$, $B(n, n)$ and $Spl(K_{1,n})$ admits prime labeling.

AMS Subject Classification: 05C78

Keywords: Prime labeling, Shadow graph, Star graph, Bistar graph, Split graph

1. INTRODUCTION

In this research manuscript we deal with graphs that are finite, connected, not directed, with no loops and multiple edges. We indicate the set of vertices by $V(G)$ and the set of edges by $E(G)$. The cardinality of the vertex set is denoted by $|V(G)|$. Labeling of graphs has significant applications in various fields like the study of the physical universe, telecommunication network, circuit design, manage data in a database and much more. The concept of Prime labeling was established by R. Entringer. A graph $G = (V(G), E(G))$ is observed to admit prime labeling when the vertices of the graph are labeled with unique integral values from $[1, |V|]$ in a way that for every edge $uv$ the labels designated to $u$ and $v$ share no common positive factors except 1[4]. A prime graph is the one that accepts a prime labeling. In our previous works we have investigated the prime labeling of 2-shadow and 3-shadow graphs. In this research article we generalize the result and we prove that the $m$-Shadow graphs of some star related graphs such as star graph $K_{1,n}$, Bistar graph $B(n, n)$ and Splitting star graph $Spl(K_{1,n})$ admits Prime labeling.

2. PRELIMINARIES

**Definition 2.1.**

Consider a bipartite graph with partitions $V_1 = \{u\} \text{ and } V_2 = \{u_1, u_2, u_3, \ldots, u_m\}$ such that the vertex $u$ of $V_1$ is adjacent to every vertex of the partition $V_2$. Such a graph is known as the star graph $K_{1,n}$ [2].

**Definition 2.2.**

Consider a star graph $K_{1,n}$ with partitions $V_1 = \{u\} \text{ and } V_2 = \{u_1, u_2, u_3, \ldots, u_m\}$ and its facsimile copy with partitions $U_1 = \{v\} \text{ and } U_2 = \{v_1, v_2, v_3, \ldots, v_n\}$. A bistar graph is attained by joining the vertex $u$ of $V_1$ with the vertex $v$ of $U_1$ by an edge [3].

**Definition 2.3.**

Consider a graph $G$. The splitting graph $Spl(G)$ of the graph $G$ is established by affixing a new vertex $v_i$ corresponding to each vertex $u_i$ of $G$ in a way that the vertices adjacent to $u_i$ are now adjacent to $v_i$ in $Spl(G)$ [1].

**Definition 2.4.**
The m-shadow graph $D_m(G)$ of a connected graph $G$ is constructed by taking $m$ facsimile copies of $G$, say $G_1$, $G_2$, $G_3$, ..., $G_m$ and then joining each vertex $u$ in $G_i$ to the neighbours of the corresponding vertex $v$ in $G_j$, $1 \leq i, j \leq m$ [5].

3. RESULTS

3.1. $m$-Shadow Graph $D_m(K_{1,n})$ of Star Graph $K_{1,n}$

**Theorem 3.1.**

$m$ - Shadow graph $D_m(K_{1,n})$ of star graph $K_{1,n}$ admits prime labeling for $n \geq m$ and $m \geq 2$.

**Proof:**

Let $G$ denote a Star graph $K_{1,n}$.

Let $G_1$, $G_2$, $G_3$, ..., $G_m$ denote $m$- facsimile copies of $G$.

In general, $\{G_k \mid 1 \leq k \leq m\}$ denote $k$th- facsimile copy of $G$.

Let $V(G_k) = \{u_k, v_i \mid 1 \leq k \leq m, (k-1)n + 1 \leq i \leq kn\}$ constitute the vertex set of $G_k$.

Let $E(G_k) = \{u_kv_i \mid 1 \leq k \leq m, (k-1)n + 1 \leq i \leq kn\}$ constitute the edge set of $G_k$.

Let $D_m(G)$ be the $m$- Shadow graph of $G$.

$V[D_m(G)] = \{u_k, v_i \mid 1 \leq k \leq m, 1 \leq i \leq kn\}$ is the vertex set of $D_m(G)$.

$E[D_m(G)] = \{u_kv_i \mid 1 \leq k \leq m, 1 \leq i \leq kn\}$ is the edge set of $D_m(G)$.

$|V[D_m(G)]| = m(n+1)$.

We define a bijective function,

$f: V[D_m(G)] \rightarrow \{1,2,3,...,m(n+1)\}$ given by,

$f(u_k) = 1, f(u_k) = p_{k-1}$ for $2 \leq k \leq m$

wherein $p_1$ is the highest prime less than the number $m(n+1)$.

$p_2$ is the second highest prime less than the number $m(n+1)$ and $p_2 < p_1$.

$p_3$ is the third highest prime less than the number $m(n+1)$ and $p_3 < p_2 < p_1$.

$\cdots$

$p_{k-1}$ is the $(k-1)$th highest prime less than the number $m(n+1)$ and

$p_{k-1} < p_{k-2} < p_{k-3} < \cdots < p_3 < p_2 < p_1$.

For all $1 \leq i \leq m(n+1)$, where $f(v_i) \neq p_j$ for $1 \leq j \leq m - 1$, we define
\[
f(v_i) = \begin{cases} 
  i + 1, & 1 \leq i \leq p_{m-1} - 2 \\
  i + 2, & p_{m-1} - 1 \leq i \leq p_{m-2} - 3 \\
  i + 3, & p_{m-2} - 2 \leq i \leq p_{m-3} - 4 \\
  i + 4, & p_{m-3} - 3 \leq i \leq p_{m-4} - 5 \\
  \vdots \\
  i + k, & p_{m-k+1} - (k - 1) \leq i \leq p_{m-1} - (k + 1) \\
  \vdots \\
  i + m, & p_1 - (m - 1) \leq i \leq m(n + 1) 
\end{cases}
\]

For the edges \( u_kv_i, 1 \leq k \leq m, 1 \leq i \leq kn \)
\[
gcd(f(u_k), f(v_i)) = \begin{cases} 
  \gcd(1, f(v_i)) = 1 \\
  \gcd(1, i + 2) = 1 \text{ if } q - 1 \leq i \leq p - 3 \\
  \gcd(1, i + 3) = 1 \text{ if } p - 2 \leq i \leq 3n 
\end{cases}
\]

Therefore, the Shadow graph \( D_m(K_{1,n}) \) of star graph \( K_{1,n} \) is a prime graph for \( n \geq m \) and \( m \geq 2 \).

Illustration 3.1.1.

![Figure 2](image)

Figure 2. Prime Labeling of 7- Shadow graph \( D_7(K_{1,7}) \) of star graph \( K_{1,7} \)

3.2. \( m \)-Shadow Graph \( D_m(B(n, n)) \) of Bistar Graph \( B(n, n) \)

Theorem 3.2.

\( m \)-Shadow graph \( D_m(B(n, n)) \) of bistar graph \( B(n, n) \) is a prime graph for \( n \geq m \) and \( m \geq 2 \).

Proof:
Let $G$ denote Bistar graph $B(n,n)$, $n \geq 2$.

Let $G_1, G_2, G_3, \ldots, G_m$ denote $m$-facsimile copies of $G$.

In general, $\{G_k / 1 \leq k \leq m\}$ denote $k$th-facsimile copy of $G$.

Let $V(G_k) = \{u_i, v_i, u_i, v_i / 1 \leq k \leq m, (k-1)n + 1 \leq i \leq kn\}$ constitute the vertex set of $G_k$.

Let $E(G_k) = \{u_k u_i / 1 \leq k \leq m, (k-1)n + 1 \leq i \leq kn\}$ and

$\{v_k'v_i / 1 \leq k \leq m, (k-1)n + 1 \leq i \leq kn\} \cup \{u_k'v_i / 1 \leq k \leq m, (k-1)n + 1 \leq i \leq kn\}$ constitute the edge set of $G_k$.

Let $D_m(G)$ be the $m$-Shadow graph of $G$.

$V[D_m(G)] = \{u_k', v_k', u_i, v_i / 1 \leq k \leq m, 1 \leq i \leq kn\}$ is the vertex set of $D_m(G)$.

$E[D_m(G)] = \{u_k u_i / 1 \leq k \leq m, 1 \leq i \leq kn\} \cup \{v_k v_i / 1 \leq k \leq m, 1 \leq i \leq kn\} \cup \{u_k' v_i / 1 \leq k \leq m, 1 \leq i \leq kn\}$ is the edge set of $D_m(G)$.

$|V[D_m(G)]| = 2m(n + 1)$.

We define a bijective function,

$f: V[D_m(G)] \rightarrow \{1, 2, 3, \ldots, 2m(n + 1)\}$ given by,

$f(u_i') = 1$, $f(u_k') = p_{k-1}$ for $2 \leq k \leq m$

Wherein, $p_1$ is the highest prime less than the number $m(n + 1)$

$p_2$ is the second highest prime less than the number $m(n + 1)$ and $p_2 < p_1$.

$p_3$ is the third highest prime less than the number $m(n + 1)$ and $p_3 < p_2 < p_1$.

\ldots

$p_{k-1}$ is the $(k-1)$th highest prime less than the number $m(n + 1)$ and

$p_{k-1} < p_{k-2} < p_{k-3} < \ldots < p_3 < p_2 < p_1$.

$$
f(v_i') = 2^i, 1 \leq i \leq m
$$

$$
f(u_i) = \begin{cases} 
2i + 2j, & 2^{i-1} - j + 1 \leq i \leq 2^j - j - 1 \text{ where } j = 2, 3, 4, \ldots m \\
2i + 2m, & 2^{i-1} - j + 1 \leq i \leq mn \text{ for } m \geq 2 \text{ and } n \geq m
\end{cases}
$$

$$
f(v_i) = 2j + 1, \text{ where } j = 1, 2, 3 \ldots \text{ and } 2j + 1 \neq p_{k-1} \forall 2 \leq k \leq m
$$

(i.e., $f(v_i)$ maps all odd numbers from 3 to $2m(n+1)$-1 except for $p_{k-1} \forall 2 \leq k \leq m$)

We observe that,

For $1 \leq i \leq mn$, $1 \leq k \leq m$.

For the edges $u_k' u_i$, $\text{gcd}(f(u_k'), f(u_i)) = 1$

For the edges $v_k' v_i$, $\text{gcd}(f(v_k'), f(v_i)) = 1$
For the edges $u_k'v_k'$, $\gcd(f(u_k'), f(v_k')) = 1$

Therefore, the Shadow graph $D_m(B(n, n))$ of bistar graph $B(n, n)$ is a prime graph for $n \geq m$ and $m \geq 2$

Illustration 3.2.1.

Figure 5. 3-Shadow graph $D_3(B(9,9))$ of bistar graph $B(9,9)$

3.3. $m$-Shadow graph $D_m(Spl(K_{1,n}))$ of Splitting Star Graph $Spl(K_{1,n})$

Theorem 3.3.

$m$-Shadow graph $D_m(Spl(K_{1,n}))$ of splitting star graph $Spl(K_{1,n})$ admits prime labeling for $n \geq m$ and $m \geq 2$.

Proof:

Let $G$ denote a splitting star graph $Spl(K_{1,n})$. Let $G_1, G_2, G_3, \ldots, G_m$ denote $m$-facsimile copies of $G$.

In general, $\{ G_k / 1 \leq k \leq m \}$ denote $k$th-facsimile copy of $G$.

Let $V(G_k) = \{ u_k'v_k', u_i, v_i / 1 \leq k \leq m, (k-1)n+1 \leq i \leq kn \}$ constitute the vertex set of $G_k$.

Let $E(G_k) = \{ u_k'v_k', u_i, v_i / 1 \leq k \leq m, (k-1)n+1 \leq i \leq kn \}$ union $\{ u_k'v_i / 1 \leq k \leq m, (k-1)n+1 \leq i \leq kn \}$ union $\{ v_k'u_i / 1 \leq k \leq m, (k-1)n+1 \leq i \leq kn \}$ constitute the edge set of $G_k$.

Let $D_m(G)$ be the $m$-Shadow graph of $G$.

$V[D_m(G)] = \{ u_k'v_k', u_i, v_i / 1 \leq k \leq m, 1 \leq i \leq kn \}$ is the vertex set of $D_m(G)$.

$E[D_m(G)] = \{ u_k'u_i / 1 \leq k \leq m, 1 \leq i \leq kn \}$ union $\{ u_k'v_i / 1 \leq k \leq m, 1 \leq i \leq kn \}$ union $\{ v_k'u_i / 1 \leq k \leq m, 1 \leq i \leq kn \}$ is the edge set of $D_m(G)$.
We define a bijective function, 
\[ f: V[D_m(G')] \to \{1, 2, 3, \ldots, 2m(n+1)\} \]
given by, 
\[ f(u'_k) = 1, f(u'_k) = p_{k-1} \text{ for } 2 \leq k \leq m \]
Wherein, \( p_1 \) is the highest prime less than the number \( m(n+1) \)
\( p_2 \) is the second highest prime less than the number \( m(n+1) \) and \( p_2 < p_1 \).
\( p_3 \) is the third highest prime less than the number \( m(n+1) \) and \( p_3 < p_2 < p_1 \).
\[ \ldots \]
\( p_{k-1} \) is the \( (k-1) \) th highest prime less than the number \( m(n+1) \) and
\[ p_{k-1} < p_{k-2} < p_{k-3} < \cdots < p_3 < p_2 < p_1 \].
\[ f(v'_i) = 2^i, 1 \leq i \leq m \]
\[ f(u_i) = \begin{cases} 
2i + 2j, & 2j - 1 < i \leq 2j - 1 \text{ where } j = 2, 3, 4, \ldots, m \\
2i + 2m, & 2m - 1 < i \leq mn \text{ for } m \geq 2 \text{ and } n \geq m 
\end{cases} \]
\[ f(v_i) = 2j + 1, \text{ where } j = 1, 2, 3, \ldots, 2j + 1 \neq p_{k-1} \forall 2 \leq k \leq m \]
(i.e., \( f(v_i) \) maps all odd numbers from 3 to \( 2m(n+1) - 1 \) except for \( p_{k-1} \forall 2 \leq k \leq m \)).

We observe that,

For \( 1 \leq i \leq mn, 1 \leq k \leq m \)
For the edges \( u'_k u_i \), \( \gcd(f(u'_k), f(u_i)) = 1 \)
For the edges \( u'_k v_j \), \( \gcd(f(u'_k), f(v_j)) = 1 \)
For the edges \( v'_k v_j \), \( \gcd(f(v'_k), f(v_j)) = 1 \)
Therefore, the Shadow graph \( D_m(Spl(K_{1,n})) \) of splitting star graph \( Spl(K_{1,n}) \) is a prime graph for \( n \geq m \) and \( m \geq 2 \).

Illustration 3.3.1

![Figure 7.4-Shadow graph $D_4(Spl(K_{1,4}))$ of splitting star graph $Spl(K_{1,4})$](image-url)
Conclusion

In this research article we conclude that the \(m\)-Shadow graph of some star related graphs such as Star graph \(K_{1,n}\), Bistar graph \(B(n,n)\) and Splitting star graph \(Spl(K_{1,n})\) admit Prime Labeling for \(n \geq m\) and \(m \geq 2\). For \(n < m\), these graphs admit prime labeling for lesser values of \(m\) and as the value of \(m\) increases they do not admit prime labeling. For instance, \(D_3(K_{1,2})\), \(D_4(B(2,2))\), \(D_4(Spl(K_{1,2}))\) are prime graphs but \(D_{10}(K_{1,2})\), \(D_5(B(2,2))\), \(D_6(Spl(K_{1,3}))\) are not prime graphs. Hence in general we conclude that Star graph \(K_{1,n}\), Bistar graph \(B(n,n)\) and Splitting star graph \(Spl(K_{1,n})\) admit Prime Labeling for \(n \geq m\) and \(m \geq 2\). The study of \(m\)-shadow graph of Star graph, Bistar graph and splitting star graph which can be used for future study and research work as it is very gripping for inspection and exploration in the field of graph theory.

References


