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A New Robust Fuzzy Informative Standard Bayes Estimator for Exponential Distribution

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Abstract:

In this paper , a method is proposed to transform any traditional probability distribution into a fuzzy probability distribution using the α-cut set principle, by finding the fuzzy cumulative distribution function $\tilde{F}(\tilde{t}_{A(\alpha)})$ at any value of the set of segments $\tilde{t}_{A(\alpha)}$ and then derive this fuzzy cumulative to find the fuzzy probability function $\tilde{f}(\tilde{t})$ And then to find a Bayesian method that has robust by proposing that for each of the parameters to be estimated at each item of the sample vector t_i drawn from a probability distribution $\varphi(t_i|\,\theta_i)$ there is an initial information represented by an initial distribution $\pi(\theta_i|\,\underline{\vartheta})$ for the parameter θ_i with the meta parameter(s) $\underline{\vartheta}$ By integrating the proposed fuzzy probability distribution with the proposed robust Bayesian method, a new general robust Bayesian fuzzy method was obtained for the probability distributions. By using Monte-Carlo simulation experiments, the proposed method was tested and compared with the standard Bayesian method. It was concluded that the proposed method is effective in estimating the parameters of the exponential distribution more accurately than the traditional Bayesian method when the data contains outliers

Keywords: Exponential distribution, Bayesian estimation, prior distribution, fuzzy logic, membership function, robustness

1. Introduction

Robust statistic is an extension of classic statistic that specifically takes into account the fact that traditional models only provide an approximation of the true basic random mechanism that generates the data. But in practice, the model assumptions are almost completely incompatible with what this random mechanism offers. It can be part of the observations that have patterns that do not share with the bulk of the rest of the data and therefore be outliers. The occurrence of deviations from the model assumptions with atypical values may have unexpected and bad effects on the results of the analysis. If we deal with the concept of robust from the point of view of Bayes theory, we will find that it depends on three main trends, the first depends on the inaccuracy of previous information (Priors) , and the second depends on the contamination of the current sample observations or previous observations or the failure to achieve hypotheses of random errors, while the last trend is based on inaccuracy in determining the loss function. The issue of robust estimates in the context of inference is one of the important issues. In (1853) Box put forward the idea of robustness and said that to build an effective model, it must be robust to ensure that there are no risks in it and thus lead to reliable and reliable inferences (Passarin, 2004 ,1). The two Azerbaijani scholars (Lotfi Zadah) and German (D. Klaua) were the first to lay the foundations of the fuzzy sets theory in (1965) when they used the term fuzzy variables on approximate, inaccurate or undefined linguistic expressions and expressions. The fuzzy set is a set of elements in which each element has a degree of affiliation between zero and one that distinguishes it from other elements in the set. It is determined by an affiliation function. (Zadeh, 1965) (Klaua, D., 1965). The researchers (Berger & Berliner) in (1985) were the first to use the idea of the robust Bayesian estimation from two sides, the first depended on the pollution class –contamination ε by defining different pollution rates in the data, and the second relied on the class of maximum likelihood, the second type ML-II for the normal distribution using simulation Monte Carlo. (Berger & Berliner, 1985). After that, it followed many studies and research that dealt with the issue of fuzzy and the issue of robust Bayesian, In 2010, (Karpisek and others) relied on the fuzzy probability distribution and its properties to define the fuzzy reliability, as they described two models of fuzzy reliability using the Fuzzy Weibull distribution to estimate the fuzzy reliability of concrete structures (Karpisek & et al, 2010), Also. (Kareema) and (Abdul Hameed) (2012) derived the fuzzy probability mass function of the geometric distribution, the fuzzy cumulative distribution function, and some properties of the fuzzy distribution such as the fuzzy mean, the fuzzy variance, and the generation of fuzzy moments. The parameter domain, as well as all formulas that use probability theory, can be fuzzy. (Kareema, 2012 & Abdul Hameed). In 2014, (Safdar) presented a new method for obtaining a fuzzy probability distribution based on the well-known probability density function of the distribution and based on the (Resolution-Identity) property to obtain a fuzzy number and proved the effectiveness and adequacy of this method. (Safdar,2014). In 2002, (Adam) compared the Bayesian decision theory in the absence of the robust decision theory with the Bayesian decision theory with the presence of the robust decision theory (Adam, 2020). In 2018, (Wang & Beli) proposed a robust Bayesian model as an alternative to the standard model that gives protection for data that include outlier values or move away from basic assumptions (Wang &Beli,2018). In 2016, Seo & Kim) proposed a robust Bayesian method based on the robust of the prior distribution of the parameters of the exponential distribution with two parameters in light of the first type hybrid control data, and for each parameter the corresponding subsequent distribution of the robust Bayes estimator was derived under a squared error loss function, and through simulation experiments the method was tested on a real data set using standard mean squares error and the amount of bias (Seo & Kim, 2016). In 2019, (Panwar) and others used the robust Bayesian approach to analyze life-times of the Maxwell distribution based on the prior distribution, the class of maximum likelihood, the second type, under a square loss function and a Linux loss function in the case of complete data and data Type I progressive hybrid control (Panwar et al,2019). In the year 2020, (Shan) and others presented a method for estimating the partial linear regression model using the Bayesian method when assuming a departure from the normal distribution and it was compared with the traditional methods. (Shan et al, 2020).

2. Crisp and fuzzy set

Let Ω is Universe of discourse , A subset from it , then each element in A may be belonging or not belonging to A. (H. Garg et al, 2013, 397) (A. Ibrahim, A. Mohammed, 2017, 143)
Let $\mu_A(x)$ is a characteristic function for A give the membership in Ω to A, it is a binary function, {0, 1}, where,

$$\mu_A(x) = \begin{cases} 1, & if\ x \in A \\ 0, & if\ x \notin A \end{cases}$$

If $\mu_A(x)$ = 1, then the element x has full belonging to the set A. If $\mu_A(x)$= 0, then the element x does not belong to the set A. Figure (1) shows the crisp set, as we note in it that belonging to the elements x_r and x_{r+1}equals zero and to the elements x_0 , x_1 x_2 equal to one, and that the elements in it either belong to the set or do not belong to it.

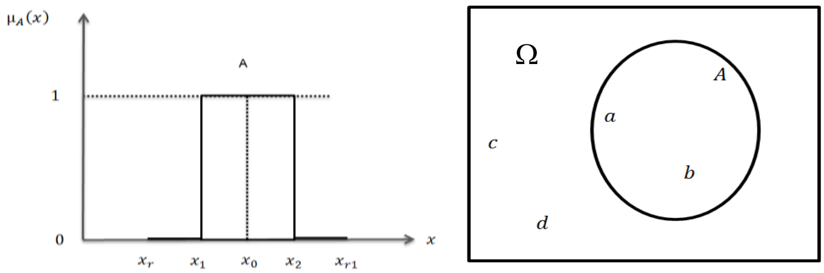

Figure (1) graphical representation of the Crisp set

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As for the fuzzy set, it is a set of ambiguous boundaries, each element in the fuzzy set has a certain degree of membership, and the fuzzy set is characterized by a membership function that assigns each element in the set a degree of membership in the interval [0, 1]. In which the element or object is allowed to belong partly. (Pak, 2017, 504)

Let Ω is Universe of discourse , a fuzzy subset $\widetilde{A}$ from it that distinguished with the membership function $\mu_{\widetilde{A}}(x)$ which produce values in the interval [0, 1] for each values in the fuzzy sample space, then the fuzzy set is , (Danyaro & et al., 2010, 240)

$$\widetilde{A} = \{(x_i, \mu_{\widetilde{A}}(x_i)), x \in \Omega, i = 1,2,3, \dots \dots n, 0 < \mu_{\widetilde{A}}(x) < 1\} \qquad \dots (1)$$

Figure (3) shows the fuzzy set, as we note in it that the membership to the elements a, c can fall between zero and one, and the element b has a degree of membership equal to one, and that the elements can belong to set A with different degrees of membership.

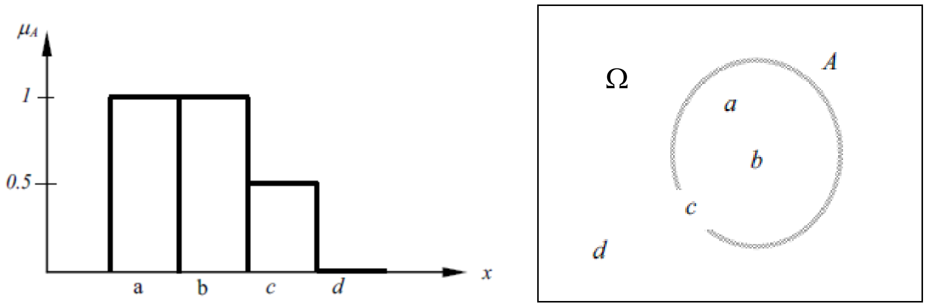

Figure (2) graphic representation of the fuzzy set

3. Suggested Fuzzy Probability Distribution

Let we have a failure time $t_1, t_2, \dots, t_n$ where $t \in T$ inaccurate, uncertain, and expressed in fuzzy numbers $\tilde{t} \in \widetilde{T}$, where $\tilde{t} = \{[0, \infty), \mu_{\tilde{t}}(t)\}$. The crisp sample observations vector that we can get from the fuzzy set, which represents all the elements that have a degree of membership greater or equal to the alpha-cut (α-cut), which represents the degree of membership of the elements we are interested in and expresses those elements as the set $A^{(\alpha)}$

$$A^{(\alpha)} = \{\tilde{t} = [0, \infty) \in \widetilde{T}, \mu_{\tilde{t}}(t) = \alpha ;\ \mu_{\tilde{t}}(t) \geq \alpha\} \quad , 0 < \alpha < 1 \qquad \dots \quad (2)$$

$\mu_{\tilde{t}}(t)$ is a membership function through which a degree of membership is generated for each failure time in the sample space and can take any form of membership functions, then $\tilde{t}_{A^{(\alpha)}}$ is Borel Measurable which will represent the fuzzy sample space and the events represent the smallest sigma-borel field (σ-Borel).Then the fuzzy cumulative distribution function $\widetilde{CDF}$ ism=,

$$\tilde{F}(\tilde{t}_{A^{(\alpha)}}) = \int_0^{\tilde{t}_{A^{(\alpha)}}} f(u) du \qquad \dots (3)$$

By deriving the equation (32-2) for ($\tilde{t}_{A^{(\alpha)}}$) we get the fuzzy probability distribution as follows:

$$\tilde{f}(\tilde{t}) = \frac{\partial \tilde{F}(\tilde{t}_{A^{(\alpha)}})}{\partial \tilde{t}_{A^{(\alpha)}}} = \frac{\partial}{\partial \tilde{t}_{A^{(\alpha)}}} \left[\int_0^{\tilde{t}_{A^{(\alpha)}}} f(u) du\right] \ ;\ 0 < \tilde{t}_{A^{(\alpha)}} < \infty \qquad \dots (4)$$

4. Fuzzy exponential distribution:

The probability density function for a crisp exponential distribution is:

$$f(t, \lambda) = \lambda e^{-\lambda t} \quad ; \quad t > 0 \qquad \dots (5)$$

From (4) we obtain,

$$\tilde{F}(\tilde{t}_{A^{(\alpha)}}) = \int_0^{\tilde{t}_{A^{(\alpha)}}} f(u) du$$

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$$= \int_0^{\tilde{t}_{A(\alpha)}} \lambda e^{-\lambda u}\, du$$

$$= \lambda \int_0^{\tilde{t}_{A(\alpha)}} e^{-e^{-\lambda u}} du$$

$$= -e^{-\lambda u}\Big]_0^{\tilde{t}_{A(\alpha)}}$$

$$= 1 - e^{\lambda \tilde{t}_{A(\alpha)}} = F(\tilde{t}_{A(\alpha)}) \qquad \ldots (6)$$

The probability density function for the fuzzy exponential distribution can be obtained as follows:

$$\tilde{f}(\tilde{t}_{A(\alpha)}) = \frac{\partial \tilde{F}(\tilde{t}_{A(\alpha)})}{\partial \tilde{t}_{A(\alpha)}} = \frac{\partial}{\partial \tilde{t}}\left[1 - e^{-\lambda \tilde{t}_{A(\alpha)}}\right]$$

$$= \lambda e^{-\lambda \tilde{t}_{A(\alpha)}} = f(\tilde{t}_{A(\alpha)}) \qquad \ldots (7)$$

5. Proposed Robust Bayesian method

Bayesian modeling takes into account the inaccuracy of the unknown parameters in a statistical model (Gelman et al., 2014). Therefore, the Bayesian model uses a set of sample data t_i Which is represented by the likelihood function of the current observations, as we have the original distribution of the items of the current sample, which represents the probability density function of the data $\varphi(t_i/\underline{\theta})$ with parameter vector $\underline{\theta}$ and prior distribution $\pi(\underline{\theta}/\underline{\vartheta})$ with hyper-parameters $\underline{\vartheta}$.

$$\{t_i/\ \underline{\theta} \sim iid\ \varphi(t_i/\underline{\theta})\ ,\underline{\theta} \sim \pi(\underline{\theta}/\underline{\vartheta})\}, \qquad i = 1,2, \ldots \ldots, n \qquad \ldots (8)$$

To find the Joint posterior distribution,

$$h(\underline{\theta}/t_i/\underline{\vartheta}) = \frac{\pi(\underline{\theta}/\underline{\vartheta}) \prod_{i=1}^{n} \varphi(t_i/\underline{\theta})}{\int_{\forall \underline{\theta}} \pi(\underline{\theta}/\underline{\vartheta}) \prod_{i=1}^{n} \varphi(t_i/\underline{\theta})} \qquad \ldots (9)$$

We note in Model (2-68) that for the parameter estimated from the observations of the sample as a whole, there is one primary distribution, which is $\pi(\underline{\theta}/\underline{\vartheta})$ with hyper- parameters $\underline{\vartheta}$ his does not achieve robustness in the estimation because all the items of the current sample data will have a common initial distribution so that the vocabulary of the same format and the abnormal vocabulary will have the same previous probability. In order to make the model (68-2) enjoy robustness, we will suggest that for each of the parameters to be estimated at each item of the sample vector t_i drawing from $\varphi(t_i/\theta_i)$ there is preliminary information represented by an initial distribution $\pi(\theta_i/\underline{\vartheta})$ for parameter θ_i with hyper- parameters $\underline{\vartheta}$,

$$t_i/\theta_i \sim iid\ \varphi(t_i/\theta_i)\ ,\theta_i \sim iid\ \pi(\theta_i/\underline{\vartheta})\ , \qquad i = 1,2, \ldots \ldots, n \qquad \ldots (10)$$

The robust posterior joint distribution of $(\underline{\theta}/t_i)$ with parameters $\underline{\theta} = (\theta_1, \theta_2 \ldots., \theta_n)$ as following:

$$\mathbb{H}(\underline{\theta}/t_i/\underline{\vartheta}) = \frac{\prod_{i=1}^{n} \pi(\theta_i/\underline{\vartheta})\ \varphi(t_i/\theta_i)}{\int_{\forall \theta_i} \prod_{i=1}^{n} \pi(\theta_i/\underline{\vartheta})\ \varphi(t_i/\theta_i)} \qquad \ldots (11)$$

And the model (11) will include that each observation of the sample is completely independent from the other observation and is conditional on the estimation of the parameter (θ_i). In other words, the sample data will be completely independent of each other.

The probability for each of the independent and identically distributed data (iid) can be obtained as follows:

$$\varphi(t_i/\underline{\vartheta}) = \int \pi(\theta_i/\underline{\vartheta})\ \varphi(t_i/\theta_i) d\theta_i \qquad \ldots (12)$$

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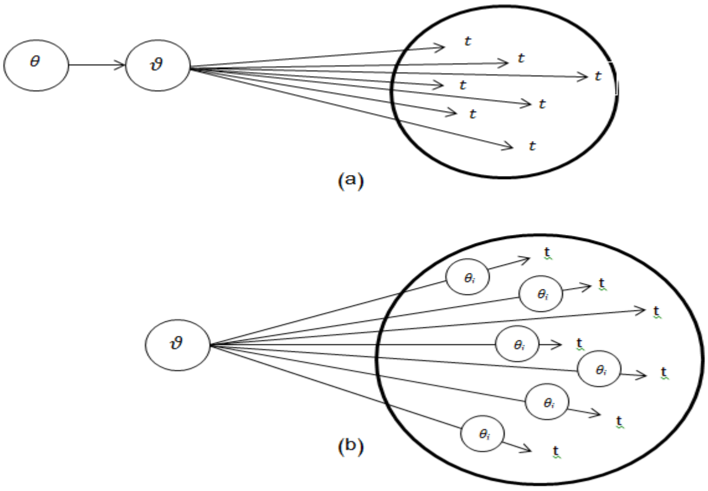

Figure (3) Graphi ayes model

6. General Formula of Proposed Fuzzy Robust Bayesian method

When we substitute the fuzzy probability distribution in the formula (7) instead of the traditional probability distribution in the proposed robust Bays formula (11), we get the following:

$$\tilde{\boldsymbol{h}}\left(\underline{\boldsymbol{\theta}}/\tilde{\boldsymbol{t}}_{A(\alpha)_i}/\hat{\boldsymbol{\vartheta}}\right)=\frac{\prod_{i=1}^{n}\pi(\theta_i/\hat{\vartheta})\,\tilde{\varphi}\left(\tilde{t}_{A(\alpha)_i}/\theta_i\right)}{\int\prod_{i=1}^{n}\pi(\theta_i/\hat{\vartheta})\,\tilde{\varphi}\left(\tilde{t}_{A(\alpha)_i}/\theta_i\right)d\theta_i} \qquad \textbf{… (13)}$$

And the formula (13) represents the fuzzy robust posterior probability distribution of the fuzzy sample data from which the fuzzy robust Bayes estimator $\underline{\hat{\hat{\theta}}}_{BRF}$ can be found at any loss function.

7. Informative standard Bayes for crisp set:

Let we have a failure time $t_1, t_2, \dots, t_n$ where $t \in T$ from exponential distribution with the probability density function:

$$f(t)=\lambda e^{-\lambda t} \qquad \dots (14)$$

Then the likelihood function is:

$$L_{exp}=\prod_{i=1}^{n} f(t_i)$$
$$=\lambda^{n} e^{-\lambda\sum_{i=1}^{n} t_i} \qquad \dots (15)$$

Suppose that there is prior information about the parameter to be estimated λ, which is represented by the probability density function of the gamma distribution with the hyper parameters a, b, which are as follows:

$$\pi(\lambda)=\frac{b^{a}}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda} \qquad \dots (16)$$

Then the joint distribution of t, λ is :

$$G(t_i,\lambda)=\frac{b^{a}}{\Gamma(a)}\lambda^{n+a-1}e^{-\lambda(\sum_{i=1}^{n} t_i+b)} \qquad \dots (17)$$

From (17), the marginal function for t_i is :

$$M(t_i)=\int_{o}^{\infty}\frac{b^{a}}{\Gamma(a)}\lambda^{n+a-1}e^{-\lambda(\sum_{i=1}^{n} t_i+b)}=\frac{b^{a}}{\Gamma(a)}\left(\frac{1}{(\sum_{i=1}^{n} t_i+b)}\right)^{n+a}\Gamma(n+a) \qquad \dots (18)$$

Then the fuzzy conditional posterior distribution as following:

$$h(\lambda|\, t_i) = \frac{G(t_i,\lambda)}{M(t_i)}$$

$$= \frac{\left(\sum_{i=1}^{n} t_i+b\right)^{n+a}}{\Gamma(n+a)} \lambda^{n+a-1} e^{-\lambda\left(\sum_{i=1}^{n} t_i+b\right)} \qquad \dots (19)$$

Which is gamma distribution with parameters ($\alpha = n + a, \beta = \sum_{i=1}^{n} t_i + b$)

Then the informative Bayes estimator for crisp set under squared error loss function is the expectation of posterior, then,

$$\hat{\lambda}_{INSBexp} = \frac{n+a}{\sum_{i=1}^{n} t_i+b} \qquad \dots (20)$$

The hyper parameters are supposing it a small numbers.

8. Suggested Robust Fuzzy Informative Standard Bayesian Estimator

Let we have the failure times $t_1, t_2, \dots, t_n$, $t\in T$, from exponential distribution with parameter λ , then the fuzzy set for the cut α which is $\widetilde{A}_\alpha = \{\tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_{\tilde{n}}\}$, $\tilde{t}\in\widetilde{T}$ و $\tilde{t} = \{[0,\infty), \mu_{\tilde{t}}(t)\}$, have a fuzzy exponential distribution with parameter λ with the following fuzzy probability density function,

$$\tilde{f}(\tilde{t}_{A(\alpha)}) = \lambda e^{-\lambda \tilde{t}_{A(\alpha)}} \qquad \dots (21)$$

Suppose that there is prior information about the parameter to be estimatedλ, which is represented by the probability density function of the gamma distribution with the hyper parameters a, b, which are as follows:

$$\pi(\lambda) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda} \qquad \dots (22)$$

We suggest the robust fuzzy Bayes estimation where there is a prior distribution for each parameter from each observation from the sample units as following:

$$\pi(\lambda_i) = \frac{b^a}{\Gamma(a)} {\lambda_i}^{a-1} e^{-\lambda_i b} \qquad \dots (23)$$

Then the joint distribution of $\tilde{t}_{A(\alpha)}$, λ is:

$$G(\tilde{t}_{A(\alpha)_i}, \lambda_i) = \left(\frac{b^a}{\Gamma(a)}\right)^{\tilde{n}} \prod_{i=1}^{\tilde{n}} {\lambda_i}^a e^{-\lambda_i\left(\tilde{t}_{A(\alpha)_i}+b\right)} \qquad \dots (24)$$

From equation (24), the marginal function for $\tilde{t}_{A(\alpha)_i}$ is:

$$M(\tilde{t}_{A(\alpha)_i}) = \left(\frac{b^a}{\Gamma(a)}\right)^{\tilde{n}} \prod_{i=1}^{\tilde{n}} \left(\frac{1}{\tilde{t}_{A(\alpha)_i}+b}\right)^{a+1} \Gamma(a+1) \qquad \dots (25)$$

From equation (24),

Then the fuzzy posterior distribution is:

$$h\left(\lambda|\, \tilde{t}_{A(\alpha)_i}\right) = \frac{G(\tilde{t}_{A(\alpha)_i}\, \lambda_i)}{M(\tilde{t}_{A(\alpha)_i})}$$

$$= \prod_{i=1}^{\tilde{n}} \frac{\left(\tilde{t}_{A(\alpha)_i}+b\right)^{(a+1)}}{\Gamma(a+1)} {\lambda_i}^{(a+1)-1} e^{-\lambda_i\left(\tilde{t}_{A(\alpha)_i}+b\right)} \qquad \dots (26)$$

Which is gamma distribution with parameters ($\alpha = a + 1, \beta = \tilde{t}_{A(\alpha)_i} + b$)

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Then, the suggested fuzzy robust informative Bayes estimator for crisp set under squared error loss function is the expectation of posterior, then,

$$\hat{\lambda}_{\mathrm{INRFSBexp}} = \prod_{i=1}^{\tilde{n}} \frac{a+1}{\tilde{t}_{A^{(\alpha)}i}+b} \qquad \dots (27)$$

The hyper parameters are suggesting estimate it according the maximum likelihood as following:

$$\pi(\lambda_i) = \frac{b^a}{\Gamma(a)} {\lambda_i}^{a-1} e^{-\lambda_i b}$$

$$L = \prod_{i=1}^{\tilde{n}} \pi(\lambda_i)$$

$$= \prod_{i=1}^{\tilde{n}} \frac{b^a}{\Gamma(a)} {\lambda_i}^{a-1} e^{-\lambda_i b}$$

$$= \left(\frac{b^a}{\Gamma(a)}\right)^{\tilde{n}} \prod_{i=1}^{\tilde{n}} {\lambda_i}^{a-1} e^{-\lambda_i b}$$

$$\mathrm{Ln}(L) = \tilde{n}\mathrm{Ln}\left(\frac{b^a}{\Gamma(a)}\right) + (a+1)\sum_{i=1}^{\tilde{n}} \ln(\lambda_i) - \lambda_i b$$

$$= a\tilde{n}\mathrm{Ln}(b) - \tilde{n}\ln\big(\Gamma(a)\big) + (a+1)\sum_{i=1}^{\tilde{n}} \ln(\lambda_i) - \lambda_i b \qquad \dots (28)$$

From equation (28) we derivative for a and b and equal to zero ,

$$\frac{\partial \mathrm{Ln}(L)}{\partial \hat{a}} = \tilde{n}\mathrm{Ln}\big(\hat{b}\big) - \frac{\tilde{n}}{\Gamma(\hat{a})}\psi(a) - \sum_{i=1}^{\tilde{n}} \ln(\lambda_i) = 0 \qquad \dots (29)$$

Where:

$\Gamma(\hat{a})$ is gamma function which it $\int_0^\infty {\lambda_i}^{\hat{a}-1}\, e^{-\frac{\lambda_i}{\hat{a}}}$

$\psi(a) = \Gamma'(\hat{a})$ is the first derivative for gamma function which is digamma according to Hurwitz Zeta function series.

$$\psi(a) = \sum_{i=1}^{n} \frac{1}{(a+\tilde{n})^2} \qquad \dots (20)$$

then the equation (29) result,

$$\therefore \frac{\partial \mathrm{Ln}(L)}{\partial \hat{a}} = \tilde{n}\mathrm{Ln}\big(\hat{b}\big) - \frac{\tilde{n}}{\Gamma(\hat{a})}\sum_{i=1}^{\tilde{n}} \frac{1}{(a+\tilde{n})^2} - \sum_{i=1}^{\tilde{n}} \ln(\lambda_i) = 0 \qquad \dots (31)$$

$$\frac{\partial \mathrm{Ln}(L)}{\partial \hat{b}} = \frac{\hat{a}\tilde{n}}{\hat{b}} - \lambda_i = 0$$

$$\Rightarrow \hat{b} = \frac{\hat{a}\tilde{n}}{\lambda_i} \qquad \dots (32)$$

From equation (30) we will use the numerical analysis to obtain the estimate b

The robust fuzzy informative Bayes is,

$$\hat{\lambda}_{\mathrm{INRFSBexp}} = \prod_{i=1}^{\tilde{n}} \frac{a_{\mathrm{mle}}+1}{\tilde{t}_{A^{(\alpha)}i}+b_{\mathrm{mle}}} \qquad \dots (33)$$

9. Simulation experiments

The Monte-Carlo Simulation method was adopted for the purpose of comparing the Bayes estimators for crisp data and the proposed robust fuzzy bass estimators the Exponential distribution, an informative prior at a squared error loss function. The theoretical values for the parameter of the distribution were obtained empirically from conducting several experiments and selecting the values, then the Bayes estimates were stable and gave the best results as (λ=1, 1.5, 4, 5, 8), the cisrp data was generated that the distributions represented by the vector t from each distribution by using inverse cumulative distribution function by applying the inverse transformation method. The crisp data vector has been polluted with outlier values by finding the arithmetic mean and standard deviation of the crisp sample vector and adding the outlier values to it according to the equation $\mathrm{t\_Outlier} = \mathrm{mean}(t{:}\,i) + 3(\mathrm{SD}{:}\,i)$. The crisp sample vector $\underline{\mathrm{t\_Outlier}} = (t_1, t_2, \dots, t_n)'$ is transformed from each distribution to the fuzzy by finding the degree of membership corresponding to each of the observations of the polluted crisp sample vector using a triangular membership function as follows:

$$\mu_A(t) = \begin{cases} 0 & if \;\; t < a \\ \frac{t-a}{b-a} & if \;\; a \le t \le b \\ 1 & if \;\; t > b \end{cases} \qquad \dots (34)$$

As a represents the lowest value of the observations values of the crisp sample and b represents the largest value of the observations values of the traditional sample vector, which results in us a fuzzy sample vector $\underline{\tilde{t}} = \tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_n$ includes each observation and its corresponding degree of membership which :

$$\tilde{t}_i = \{(t_i,\ \mu_A(t_1)), (t_2,\ \mu_A(t_2)), \dots., (t_{\tilde{n}},\ \mu_A(t_n))\} \qquad \dots (35)$$

After that, the fuzzy set is obtained at the cutoff α $\widetilde{\mathrm{A}}_\alpha = \{\tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_{\tilde{n}}\}$ for the studied distribution by choosing the elements in the fuzzy set that have a degree of belonging greater or equal to the cut, α that is$\tilde{A}_\alpha = \{\tilde{t} \in T;\ \mu_{\tilde{A}}(t) \ge \alpha\}$ by choosing $\alpha - \mathrm{cut} = 0.2, 0.4, 0.5, 0.7, 0.9$. The Estimation methods were compared using the mean squared error criterion (MSE) by using Matlab 2015

First: When the data contains one outlier:

Table (1) Estimation of parameters and mean square error of MSE in the crisp and proposed Bayesian methods at cutoff coefficients α-cut=0.2,0.4,0.5,0.7,0.9 and at default value of exponential distribution parameter λ=1, or one outlier.

Distribution		**Exponential**		**Best**
cut	**Method**	**Estimation**	**MSE**	
0.2	**INSB**	**1.42765**	**0.12325**	**INRFSB**
	INRFSB	**1.33245**	**0.11254**	
0.4	**INSB**	**1.31733**	**0.11384**	**INRFSB**
	INRFSB	**1.21257**	**0.10241**	
0.5	**INSB**	**1.29734**	**0.10215**	**INRFSB**
	INRFSB	**1.11455**	**0.10184**	
0.7	**INSB**	**1.19565**	**0.10114**	**INRFSB**
	INRFSB	**1.12134**	**0.07772**	
0.9	**INSB**	**1.13635**	**0.08464**	**INRFSB**
	INRFSB	**1.11176**	**0.06369**	

Table (2) Estimation of parameters and mean square error of MSE in the crisp and proposed Bayesian methods at cutoff coefficients α-cut=0.2,0.4,0.5,0.7,0.9 and at default value of exponential distribution parameter λ=1.5, or one outlier.

Distribution		**Exponential**		**Best**
cut	**Method**	**Estimation**	**MSE**	
0.2	**INSB**	**1.54975**	**0.09251**	**INRFSB**
	INRFSB	**1.73166**	**0.44597**	
0.4	**INSB**	**1.53841**	**0.23965**	**INRFSB**
	INRFSB	**1.52245**	**0.08913**	
0.5	**INSB**	**1.53135**	**0.21568**	**INRFSB**
	INRFSB	**1.52215**	**0.08824**	
0.7	**INSB**	**1.51131**	**0.02477**	**INRFSB**
	INRFSB	**1.51905**	**0.08561**	

0.9	**INSB**	**1.51115**	**0.01254**	**INRFSB**
	INRFSB	**1.51031**	**0.07461**	

Table (2) Estimation of parameters and mean square error of MSE in the crisp and proposed Bayesian methods at cutoff coefficients α-cut=0.2,0.4,0.5,0.7,0.9 and at default value of exponential distribution parameter λ=4, or one outlier.

Distribution		**Exponential**		**Best**
cut	**Method**	**Estimation**	**MSE**	
0.2	**INSB**	**4.37364**	**0.21453**	**INRFSB**
	INRFSB	**4.31093**	**0.12585**	
0.4	**INSB**	**4.28334**	**0.12785**	**INRFSB**
	INRFSB	**4.25463**	**0.11454**	
0.5	**INSB**	**4.25311**	**0.11883**	**INRFSB**
	INRFSB	**4.23451**	**0.11213**	
0.7	**INSB**	**4.22325**	**0.11326**	**INRFSB**
	INRFSB	**4.21353**	**0.10853**	
0.9	**INSB**	**4.11126**	**0.07542**	**INRFSB**
	INRFSB	**4.10533**	**0.00568**	

Second: When the data contains three outlier:

Table (4) Estimation of parameters and mean square error of MSE in the crisp and proposed Bayesian methods at cutoff coefficients α-cut=0.2,0.4,0.5,0.7,0.9 and at default value of exponential distribution parameter λ=1, or three outlier.

Distribution		**Exponential**		**Best**
cut	**Method**	**Estimation**	**MSE**	
0.2	**INSB**	**4.89544**	**3.98666**	**INRFSB**
	INRFSB	**1.12159**	**0.00219**	
0.4	**INSB**	**4.08911**	**3.12455**	**INRFSB**
	INRFSB	**1.11094**	**0.00116**	
0.5	**INSB**	**2.11136**	**1.21111**	**INRFSB**
	INRFSB	**1.102241**	**0.00031**	
0.7	**INSB**	**1.89544**	**0.38943**	**INRFSB**
	INRFSB	**1.10019**	**0.00017**	
0.9	**INSB**	**1.21675**	**0.09554**	**INRFSB**
	INRFSB	**1.09010**	**0.00011**	

Table (5) Estimation of parameters and mean square error of MSE in the crisp and proposed Bayesian methods at cutoff coefficients α-cut=0.2,0.4,0.5,0.7,0.9 and at default value of exponential distribution parameter λ=1.5, or three outlier.

Distribution		**Exponential**		**Best**
cut	**Method**	**Estimation**	**MSE**	
0.2	**INSB**	**2.99464**	**0.65788**	**INRFSB**
	INRFSB	**1.54644**	**0.11573**	
0.4	**INSB**	**2.05853**	**0.97866**	**INRFSB**
	INRFSB	**1.53966**	**0.11047**	
0.5	**INSB**	**2.05544**	**0.97336**	**INRFSB**
	INRFSB	**1.53334**	**0.11005**	
0.7	**INSB**	**2.12422**	**0.05322**	**INRFSB**
	INRFSB	**1.51075**	**0.00467**	
0.9	**INSB**	**1.59533**	**0.11343**	**INRFSB**
	INRFSB	**1.50866**	**0.00244**	

Table (6) Estimation of parameters and mean square error of MSE in the crisp and proposed Bayesian methods at cutoff coefficients α-cut=0.2,0.4,0.5,0.7,0.9 and at default value of exponential distribution parameter λ=4, or three outlier.

Distribution		**Exponential**		**Best**
cut	**Method**	**Estimation**	**MSE**	
0.2	**INSB**	**4.53224**	**2.67354**	**INRFSB**

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	INRFSB	**4.35222**	**0.04365**	
0.4	**INSB**	**4.45111**	**1.03456**	**INRFSB**
	INRFSB	**4.21455**	**0.02186**	
0.5	**INSB**	**4.43602**	**1.01354**	**INRFSB**
	INRFSB	**4.21217**	**0.02111**	
0.7	**INSB**	**4.33213**	**1.00342**	**INRFSB**
	INRFSB	**4.11356**	**0.00236**	
0.9	**INSB**	**4.22132**	**1.00113**	**INRFSB**
	INRFSB	**4.10068**	**0.00023**	

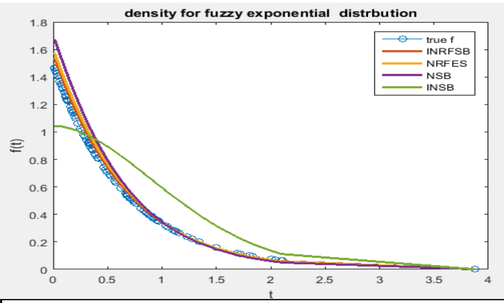

Figure (4) the curve of the probability density function for the exponential distribution at the traditional and proposed Bayesian estimation method at the cutoff 0.9

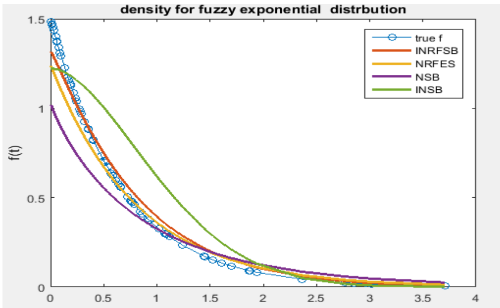

Figure (5) the curve of the probability density function for the exponential distribution at the traditional and proposed Bayesian estimation method at the cutoff 0.7

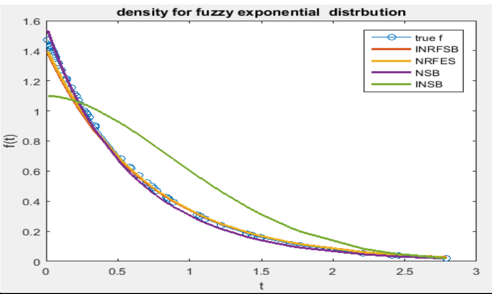

Figure (6) the curve of the probability density function for the exponential distribution at the traditional and proposed Bayesian estimation method at the cutoff 0.5

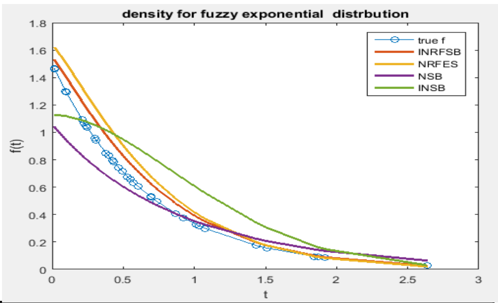

Figure (7) the curve of the probability density function for the exponential distribution at the traditional and proposed Bayesian estimation method at the cutoff 0.4

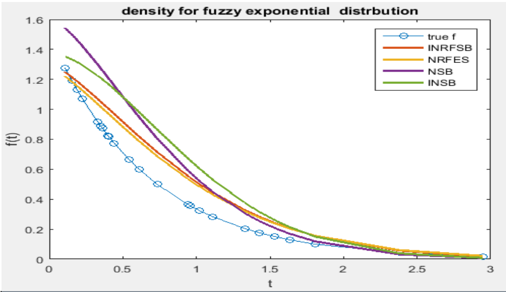

Figure (7) the curve of the probability density function for the exponential distribution at the traditional and proposed Bayesian estimation method at the cutoff 0.2

10. Results and discussion:

It is clear from Tables (1) to (6) the proposed Robust fuzzy Bayes method based on an informational prior distribution is superior to the traditional Bayes method under outliers' observations. The greater the cutoff α, the less the mean of the squares of error and the greater the accuracy of the estimates extracted according to the fuzzy robust Bayesian method and for all simulation experiments. The proposed informative robust fuzzy Bayes method was the best for the fuzzy exponential distribution.

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